MTG 5316/4302 PRACTICE FINAL EXAM

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Work all problems and show all work. Each problem is worth ten points. Partial credit will be given for correct reasoning. Credit will be deducted for statements and calculations that are not correct.

Problem 1. Suppose that $X_\alpha \neq \emptyset$ for all $\alpha \in A$. Let $\beta \in A$ and define

$$\pi_\beta : \prod_{\alpha \in A} X_\alpha \to X_\beta$$

by $\pi_\beta(x_\alpha) = x_\beta$. Show that π_β is onto for every $\beta \in A$.

Problem 2. Show that for every set X, there is no function $f : X \to 2^X$ such that f is onto.

Problem 3. Let (X, d) be a metric space and let $A, B \subset X$ be closed subsets of X such that $A \cap B = \emptyset$. Show that there is a continuous function $f : X \to [0, 1]$ such that $A \subset f^{-1}(1)$ and $B \subset f^{-1}(0)$.

Problem 4. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous. Suppose that $f([a, b]) \supset [c, d]$. Show that there is an interval $[\alpha, \beta] \subset [a, b]$ such that $f([\alpha, \beta]) = [c, d]$.

Problem 5. Let $f : [a, b] \to \mathbb{R}$ be continuous. Suppose that f has a point of period three. Show that f has a point of period 17.

Problem 6. Suppose that X is a complete metric space. A function $f : X \to X$ is said to be a *contraction mapping* provided that there is a $0 < c < 1$ such that for all $x, y \in X$, $d(f(x), f(y)) \leq c \cdot d(x, y)$. Show that if $f : X \to X$ is a contraction mapping, then there is a unique $z \in X$ such that $f(z) = z$. This is known as the **Contraction Mapping Theorem** or the **Banach Fixed Point Theorem**.

Problem 7. Suppose that X is a compact metric space. Show that X is complete.

Problem 8. Suppose that X is compact Hausdorff ($\mathbf{T}_2$). Show that X is normal.

Problem 9. (**Tietze Extension Theorem**) Suppose that X is normal ($\mathbf{T}_4$). Suppose that $A \subset X$ is closed in X. Let $f : A \to [0, 1]$ be continuous. Show that there is an $F : X \to [0, 1]$ which is continuous such that $F|_A \equiv f$.

Problem 10. Suppose that U is a connected open set in $\mathbb{R}^n$. Show that for $x, y \in U$, there is a continuous function $f : [0, 1] \to U$ such that $f(0) = x$ and $f(1) = y$.