MTG 5316/4302 PRACTICE FINAL EXAM

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Work all problems and show all work. Each problem is worth ten points. Partial credit will be given for correct reasoning. Credit will be deducted for statements and calculations that are not correct.

Problem 1. State and prove Urysohn's Lemma.

Problem 2. Let X be a topological space and let $f_n : X \to \mathbb{R}$ be continuous for each $n = 1, 2, \ldots$. Suppose that $c_n > 0$ for all $n = 1, 2, \ldots$ and that $\sum_{n=1}^{\infty} c_n < \infty$. Suppose that for each n and all $x \in X$, $|f_n(x)| < c_n$. Show that $\sum_{n=1}^{\infty} f_n(x) : X \to \mathbb{R}$ is continuous.

Problem 3. Show that there is subset $C \subset [0, 1]$ such that C is homeomorphic to the Cantor middle-third set such that $\lambda(C) > 0$ where $\lambda(A)$ is the Lebesgue measure of A.

Problem 4. Let A be a countably infinite set. Consider the countable collection of intervals $\{[0, 1]_\alpha \big| \alpha \in A$. Consider the quotient space that joins the 0's of these intervals to a single point. Show that this space is not metrizable.

Problem 5. Suppose that X is a normal space with a countable basis. Show that there is a metric ρ on X which generates the topology of X.

Problem 6. Suppose that X is a compact Hausdorff space with a countable basis. Show that X is metrizable.

Problem 7. Let R be the real line with the topology having as basis $\mathscr{T} = \{[a, b) \big| a < b \in R\}$. Show that R is a normal space.

Problem 8. Show that $R \times R$ is not normal with the topology in the previous problem. Can R be a metric space?

Problem 9. Suppose that X is a separable metric space, that is, X is metric with a countable basis. Show that there is a compact metric space C such that $X \subset C$ and X is dense in C.

Problem 10. A Lindelöf space is a topological space X such that every open cover of X has a countable subcover. Show that if X is regular and Lindelöf, then it is normal.