MTG 5316/4302 PROBLEMS FOR FINAL EXAM

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These problems are candidates for the final exam. Included in the list are those problems that were on the assignments. There are some additional problems as well that you are responsible for. You may discuss the problems with members of the class and with me. You may consult our textbook and other books. You may not read the papers of other students. You may ask about the problems in class and there will be a time of review for the final exam.

You need not prepare for these problems for the final: $\{1, 2, 20, 22, 23, 24, 44, 45, 48, 49\}$. For those graduate students who may be taking the *Topology First Year Exam*, you should know all the problems listed for that exam.

Problem 1. (Problem 1.10.7 page 67 in Munkres) Let J be a well-ordered set. A subset J_0 is said to be **inductive** if for every $\alpha \in J$

$$(S_\alpha \subset J_0) \implies \alpha \in J_0.$$

Prove the following theorem.

Theorem (The principle of transfinite induction)**.** *If J is a well-ordered set and J_0 is an inductive subset of J, then $J_0 = J$.*

Problem 2. Suppose that X_α is non-empty for every $\alpha \in A$. Show that

$$\prod_{\alpha \in A} X_\alpha \neq \emptyset.$$

Problem 3. Suppose that $X_\alpha \neq \emptyset$ for all $\alpha \in A$. Let $\beta \in A$ and define

$$\pi_\beta : \prod_{\alpha \in A} X_\alpha \to X_\beta$$

by $\pi_\beta(x_\alpha) = x_\beta$. Show that π_β is onto for every $\beta \in A$.

Problem 4. Show that for every set X, there is no function $f : X \to 2^X$ such that f is onto.

Problem 5. Let (X, d) be a metric space and let $(X, \mathscr{T})$ be the associated topological space. Show that U is an open set in $(X, \mathscr{T})$, i.e., $U \in \mathscr{T}$, if and only if whenever $\{x_i\}_{i=1}^\infty \subset X$ and $x_i \to z \in U$, then there is an N such that for all $i \geq N$, $x_i \in U$.

Problem 6. Let (X, d) be a metric space and let $(X, \mathscr{T})$ be the associated topological space. Show that $(X, \mathscr{T})$ is $\mathbf{T}_0$, $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{T}_3$, and $\mathbf{T}_4$.

Problem 7. Let (X, d) be a metric space and let $A, B \subset X$ be closed subsets of X such that $A \cap B = \emptyset$. Show that there is a continuous function $f : X \to [0, 1]$ such that $A \subset f^{-1}(1)$ and $B \subset f^{-1}(0)$.

Problem 8. Let X and Y be topological spaces. Suppose that $f : X \to Y$ is continuous and suppose that $A \subset X$ is a connected set in X. Show that $f(A) \subset Y$ is a connected set in Y.

Problem 9. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and suppose that $f([a, b]) \supset [a, b]$. Show that there is an $x \in [a, b]$ such that $f(x) = x$. Such an x is said to be a *fixed point* for f.

Problem 10. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous. Suppose that $f([a, b]) \supset [c, d]$. Show that there is an interval $[\alpha, \beta] \subset [a, b]$ such that $f([\alpha, \beta]) = [c, d]$.

Problem 11. Show that there is a continuous function $f : [0, 1] \to [0, 1] \times [0, 1]$ which is onto.

Problem 12. Let $f : [a, b] \to \mathbb{R}$ be continuous. Suppose that f has a point of period three. Show that f has points of all periods.

Problem 13. State Sharkovsky's theorem.

Problem 14. Let $f : [a, b] \to \mathbb{R}$ be continuous. Suppose that f has a point of period three. How many points of period 17 can you guarantee there will be? How many orbits of period 17? Answer the same questions for 18 instead of 17. For general n instead of 17.

Problem 15. Let X and Y be metric spaces with metrics d and ρ, respectively. Show that $f : X \to Y$ is continuous if and only if for every sequence $x_n \to z$ in X, $f(x_n) \to f(z)$ in Y.

Problem 16. Suppose that X is a metric space and $x_0 \in X$. Suppose that $h : X \to X$ is continuous and suppose that $h^n(x_0) \to z$ as $n \to \infty$. Show that $h(z) = z$. We say that z is a *fixed point* for h.

Problem 17. Suppose that X is a metric space with metric d. A *Cauchy sequence* in X is a sequence $\{x_n\}_{n=1}^{\infty}$ such that if $\varepsilon > 0$, then there is an N such that for all $n, m \geq N$, $d(x_n, x_m) < \varepsilon$. We say that X is a *complete metric space* provided that for every Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ in X, there is a $z \in X$ such that $x_n \to z$ as $n \to \infty$. Give examples of complete metric spaces. Give examples of metric spaces that are not complete.

Problem 18. Suppose that X is a complete metric space. A set $A \subset X$ is *dense* provided that for all $x \in X$ and all $\varepsilon > 0$, there is an $a \in A$ such that $d(x, a) < \varepsilon$. Show that if

$\{U_n\}_{n=1}^{\infty}$ is a sequence of open sets in X such that U_n is also dense in X for each n, then $\cap_{n=1}^{\infty} U_n$ is dense in X. This is called the **Baire Category Theorem**.

Problem 19. Suppose that X is a complete metric space. A function $f : X \to X$ is said to be a *contraction mapping* provided that there is a $0 < c < 1$ such that for all $x, y \in X$, $d(f(x), f(y)) \leq c \cdot d(x, y)$. Show that if $f : X \to X$ is a contraction mapping, then there is a unique $z \in X$ such that $f(z) = z$. This is known as the **Contraction Mapping Theorem** or the **Banach Fixed Point Theorem**.

Problem 20. Show that $\mathbb{R}$ is a complete metric space. Show that $\mathbb{R}^n$ is a complete metric space.

Problem 21. Suppose that X is a compact metric space. Show that X is complete.

Problem 22. Suppose that X is a complete metric space. Show that if X is countable and non-empty, then X has an isolated point. An *isolated point* is a point $x \in X$ such that $\{x\}$ is open in X.

Problem 23. Suppose that X is a complete metric space. Suppose that B_i is closed in X for all i and suppose that (1) for all i, $B_{i+1} \subset B_i$, (2) for all i, $\varepsilon_i > 0$ with $\text{diam} B_i < \varepsilon_i$ and such that $\varepsilon_i \to 0$ as $i \to \infty$, and (3) for all i, $B_i \neq \emptyset$. Show that $\cap_{i=1}^{\infty} B_i = \{x\}$ for some $x \in X$.

Problem 24. Suppose that X is a complete metric space which is not empty. Suppose that X has no isolated points. Show that $|X| \geq 2^{\aleph_0}$.

Problem 25. Suppose that X is a complete metric space which is not empty and has no isolated points. Show that there is a function $f : \prod_{i=1}^{\infty}\{0,1\} \to X$ which is one-to-one and continuous where $\{0,1\}$ is given the discrete topology and $\prod_{i=1}^{\infty}\{0,1\}$ is given the product topology.

Problem 26. Let X be a topological space. Let $A \subset X$ and define the *closure* of A as $\overline{A} = \cap\{C \mid C \text{ is closed in } X \text{ and } C \supset A\}$. Show that $\overline{A}$ is a closed set in X. Show also that $\overline{\overline{A}} = \overline{A}$.

Problem 27. Let X be a compact topological space. Suppose that $A \subset X$ is closed in X. Show that A is compact. Give an example of a topological space with a subset A that is compact and not closed.

Problem 28. Suppose that X is a compact Hausdorff space (i.e., $\mathbf{T}_2$). Show that if $A \subset X$ is compact, then A is closed in X.

Problem 29. Suppose that X is a compact metric space. Show that every sequence in X has a convergent subsequence.

Problem 30. Suppose that X is a metric space and suppose that every sequence in X has a convergent subsequence. Show that X is compact.

Problem 31. Suppose that X is a topological space and that $A \subset X$ is compact. Suppose that $f : X \to Y$ is continuous. Show that $f(A) \subset Y$ is compact in Y.

Problem 32. Suppose that X is a metric space and that $A \subset X$ is compact. Show that A is closed and bounded in X. A subset A of a metric space X is *bounded* provided that there is an $r > 0$ and an $x \in X$ such that $A \subset B_r(x)$. Give an example of a closed and bounded subset of a metric space that is not compact.

Problem 33. Let $\{X_\alpha\}_{\alpha \in A}$ be a collection of topological spaces. Let $\beta \in A$ and let

$$\pi_\beta : \prod_{\alpha \in A} X_\alpha \to X_\beta$$

be defined by $\pi_\beta((x_\alpha)) = x_\beta$. Show that π_β is continuous.

Problem 34. Suppose that $\{X_\alpha\}_{\alpha \in A}$ is a collection of compact spaces. Show that

$$\prod_{\alpha \in A} X_\alpha$$

is compact.

Problem 34. Suppose that X is compact Hausdorff ($\mathbf{T}_2$). Show that X is normal.

Problem 35. Suppose that $\{X_\alpha\}_{\alpha \in A}$ is a collection of compact spaces. Show that

$$\prod_{\alpha \in A} X_\alpha$$

is compact.

Problem 36. (**Urysohn Lemma**) Suppose that X is normal ($\mathbf{T}_4$). Suppose that A and B are disjoint closed subsets of X. Show that there is a continuous function $f : X \to [0,1]$ such that $f^{-1}(0) \supset A$ and $f^{-1}(1) \subset B$.

Problem 37. (**Tietze Extension Theorem**) Suppose that X is normal ($\mathbf{T}_4$). Suppose that $A \subset X$ is closed in X. Let $f : A \to [0,1]$ be continuous. Show that there is an $F : X \to [0,1]$ which is continuous such that $F|_A \equiv f$.

Problem 38. Suppose that X and Y are complete metric spaces. Show that $X \times Y$ is also complete in the metric $D((x,y),(x',y')) = \max\{d(x,y), d(x',y')\}$.

Problem 39. Alexander Subbase Theorem. Let X be a topological space with a subbase $\mathscr{B}$. Suppose that for every cover $\mathscr{V} = \{B_a\}_{a \in A}$ of X by elements of $\mathscr{B}$, there is a finite subcover $\{B_{a_1}, \ldots, B_{a_n}\}$. Then X is compact. We say that $\mathscr{B}$ is a *subbase for the* *topology* of X provided that (1) B is open for every $B \in \mathscr{B}$ and (2) for every $x \in U \subset X$ with U open, there is a finite collection $\{B_1, \ldots B_n\} \subset \mathscr{B}$ such that

$$x \in \bigcap_{i=1}^{n} B_i \subset U.$$

Problem 40. Let X be a connected metric space. Let $x, y \in X$ be distinct points. Let $\varepsilon > 0$. Show that there is a chain of open sets $\{U_1, U_2, \ldots, U_n\}$ such that $\text{diam} U_i < \varepsilon$ for each i with $x \in U_1$ and $y \in U_n$. A *chain* of sets $\{A_1, \ldots, A_n\}$ is a collection such that $A_i \cap A_j \neq \emptyset$ if and only if $|i - j| \leq 1$.

Problem 41. Suppose that U is a connected open set in $\mathbb{R}^n$. Show that for $x, y \in U$, there is a continuous function $f : [0, 1] \to U$ such that $f(0) = x$ and $f(1) = y$.

Problem 42. A *homeomorphism* is a continuous function $f : X \to Y$ which is one-to-one and onto and such that $f^{-1} : Y \to X$ is also continuous. Show that if X is a compact Hausdorff, Y is Hausdorff, and $f : X \to Y$ is continuous, one-to-one, and onto, then f is a homeomorphism.

Problem 43. Two spaces X and Y are *homeomorphic* provided there is a homeomorphism $f : X \to Y$. Show that if X and Y are homeomorphic, then X is compact if and only if Y is compact. Show that the same is true for the properties connected, normal, pathwise connected, and metrizable. For what other properties is this true?

Problem 44. Define the *Cantor set* by $C = \prod_{i=1}^{\infty}\{0, 1\}$. Define a group structure on C by $(b_i) + (c_i) = (d_i)$ by add and carry. Then C with this addition is called the *dyadic integers* or the *odometer group* and is sometimes denoted Δ_2. Show that the dyadic solenoid is the quotient space of $(\Delta_2 \times [0, 1]) \,/\, ((b_i), 0) \sim ((b_i) + (1, 0, 0, \ldots), 1))$.

Problem 45. Describe and analyze the **Smale Horseshoe Map**.

Problem 46. Consider the interval with the endpoints identified. Denote this by $A = [0, 1]/\,(0 \sim 1)$. Let $q : [0, 1] \to A$ be defined in the natural way. Show that if A has the quotient topology from the function q, then A is homeomorphic to S^1, the circle.

Problem 47. Consider $[0, 1]\times[0, 1]$. Let $A = [0, 1]\times[0, 1]/\,(\{(0, y) \sim (1, y)\} \cup \{(x, 0) \sim (x, 1)\})$. Show that A with the quotient topology from $[0, 1] \times [0, 1]$ is homeomorphic to the torus $\mathbb{T}^2$.

Problem 48. Consider $\mathbb{R}^n/\mathbb{Z}^n$ and the quotient group obtained from $\mathbb{R}^n$ with group operation given by vector addition. Let $q : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ be the quotient homomorphism. Let $\mathbb{R}^n/\mathbb{Z}^n$ have the quotient topology from the map q. Show that $\mathbb{R}^n/\mathbb{Z}^n$ is homeomorphic to the $n-$dimensional torus, $\mathbb{T}^n$.

Problem 49. Let $M = (m_{ij})$ by an $n \times n$ matrix with each $m_{ij} \in \mathbb{Z}$. Let $M : \mathbb{R}^n \to \mathbb{R}^n$ be the homomorphism defined by matrix multiplication. Suppose that $\det(M) = \pm 1$. Show that M induces a function $f_M : \mathbb{R}^n/\mathbb{Z}^n = \mathbb{T}^n \to \mathbb{T}^n$. Give a formula for f_M in terms of complex multiplication by thinking of $S^1 = \mathbb{T} = \{z \in \mathbb{C} \big| |z| = 1\}$.

Problem 50. Suppose that $\{X_\alpha\}_{\alpha \in A}$ is a collection of connected topological spaces. Show that $\prod_{\alpha \in A} X_\alpha$ is connected.

Problem 51. Suppose that $\{C_\alpha\}_{\alpha \in A}$ is a collection of connected subsets of the topological space X. Suppose that $x \in C_\alpha$ for all $\alpha \in A$. Show that $\bigcup_{\alpha \in A} C_\alpha$ is connected.

Problem 52. Suppose that C is a connected subset of the topological space X. Let $\overline{C}$ denote the closure of C. Show that if $C \subset B \subset \overline{C}$, then B is connected.