MTG 5316/4302 FALL 2018 REVIEW FINAL

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Problem 1. Define open set in a metric space X. Define what it means for a set $A \subset X$ to be connected in a metric space X.

Problem 2. Show that if a set $A \subset \mathbb{R}$ is connected, then it must be an interval.

Problem 3. Define what it means for a function $f : X \to Y$ to be continuous for X and Y metric spaces.

Problem 4. Suppose that $f : X \to Y$ is continuous with X and Y metric spaces. Show that if $A \subset X$ is connected, then $f(A) \subset Y$ is connected.

Problem 5. Show that if $A \subset \mathbb{R}$ is an interval, then it is connected.

Problem 6. State and prove the **Bolzano-Weierstrass Theorem** for the real line.

Problem 7. Define what it means to be **sequentially compact**. Show that a set $A \subset \mathbb{R}$ is sequentially compact if and only if A is closed and bounded. The **Heine-Borel Theorem** states that $A \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.

Problem 8. Define a function from the Cantor Set onto the interval $[0, 1]$. Show that this function is continuous using the $\epsilon - \delta$ definition of continuituy.

Problem 9. Define **uniform continuity**. Suppose that $f : X \to Y$ is continuous with X sequentially compact. Show that f is uniformly continuous.

Problem 10. Suppose that $f : X \to Y$ is continuous. Suppose that X is sequentially compact. Show that $f(X)$ is sequentially compact.

Problem 11. Let X be a metric space. Let $A \subset X$ be connected. State and prove the **Chain Connectedness Theorem** for A.

Problem 12. Let U be an open subset of $\mathbb{R}^n$. Show that if U is connected, then for every $a \neq b \in U$, there is a continuous $f : [0,1] \to U$ such that f is one-to-one with $f(0) = a$ and $f(1) = b$.

Problem 13. Let X be a metric space with $A \subset X$. The **closure** of A is the set $\overline{A} = \left\{x \in X \;\middle|\; \exists \{x_i\}_{i=1}^{\infty} \subset A \ni \lim_{i\to\infty} x_i = x\right\}$. Note that for any set $A \subset X$, $\overline{A}$ is closed in X. Suppose that $A \subset X$ is connected. Show that $\overline{A}$ is connected.

Problem 14. Let X be a metric space and suppose that $A_\lambda \subset X$ is connected for all $\lambda \in \Lambda$. Suppose that there is an $x_0 \in X$ such that $x_0 \in A_\lambda$ for all λ. Show that

$$\bigcup_{\lambda\in\Lambda} A_\lambda$$

is connected.

Problem 15. Let $A \subset [0,1] \times [-1,1] \subset \mathbb{R}^2$ be defined by $A = \left\{\left(x, \sin\left(\frac{1}{x}\right)\right) \;\middle|\; 0 < x \leq 1\right\} \cup \{0\} \times [-1,1]$. This is the **Topologist's Sine Curve**. Show that A is connected. Show that there is no continuous function $f : [0,1] \to A$ such that $f(0) = (1, \sin(1))$ and $f(1) = (0,0)$.

Problem 16. Let X be a metric space. Define **compactness** of X in terms of open covers of X. Show that X is compact if and only if for every collection of closed sets, $\mathscr{F} = \{F_\alpha\}_{\alpha\in A}$ if $\mathscr{F}$ has the finite intersection property, then

$$\bigcap_{\alpha\in A} F_\alpha \neq \emptyset.$$

Problem 17. Let X be a metric space. Show that X is sequentially compact if and only if X is compact.

Problem 18. Let X be a metric space. Suppose that $A \subset X$ be compact. Show that A is closed in X.

Problem 19. Let X be a compact metric space. Suppose that $f : X \to Y$ be continuous, one-to-one, and onto. Show that $f^{-1} : Y \to X$ is also continuous. In case f and f^{-1} are both continuous, f is said to be a **homeomorphism** between X and Y and X and Y are said to be **homeomorphic**.

Problem 20. Are the following pairs of spaces homeomorphic? (1) $[0,1]$ and S^1. (2) $[0,1]$ and $[0,1] \times [0,1]$. (3) $[0,1]$ and $[0,1] \cup [2,3]$. (4) (a,b) with $a < b$ and $\mathbb{R}$.

Problem 21. Let X be a metric space. Suppose that $\{A_i\}_{i=1}^{\infty}$ is a countable collection of compact connected sets in X such that $A_{i+1} \subset A_i$ for all i. Show that

$$\bigcap_{i=1}^{\infty} A_i$$

is compact and connected.

Problem 22. Show that the **dyadic solenoid** is compact and connected.

Problem 23. Let $D = \prod_{i=1}^{\infty}\{0,1\}$. Give D the metric $d\left((a_i),(b_i)\right) = \sum_{i=1}^{\infty} \frac{|a_i - b_i|}{2^i}$. Show that D is homeomorphic to the Cantor middle third set.

Problem 24. Let D be as in Problem 3. Let $\mathbb{N} = \left\{(a_i) \in \prod_{i=1}^{\infty}\{0,1\} \;\middle|\; \exists N \ a_i = 0 \ \forall i > N\right\}$. Show that for the closure of $\mathbb{N}$, $\overline{\mathbb{N}}$ is all of D. If $A \subset X$ is such that $\overline{A} = X$, we say that A is **dense** in X.

Problem 25. Let $f : X \to Y$ be an onto function with X and Y metric spaces. We say that f is a **quotient map** provided that $U \subset Y$ is open if and only if $f^{-1}(U)$ is open in X. Show that if f is a quotient map, then f is continuous. Also show that if X is a compact metric space and f is continuous, then f is a quotient map.

Problem 26. Let X be a topological space. Suppose that $f : X \to Y$ is continuous. Show that if $A \subset X$ is connected, then $f(A) \subset Y$ is connected.

Problem 27. Suppose that X is a Hausdorff space. Show that if $A \subset X$ is compact, then A is closed in X.

Problem 28. Suppose that X and Y are compact Hausdorff spaces. Suppose that $f : X \to Y$ is continuous and onto. Show that f is a quotient map,

Problem 29. Suppose that X is a compact Hausdorff space. Suppose that $\{A_\lambda\}_{\lambda \in \Lambda}$ is a **chain** of closed connected sets. Show that

$$\bigcap_{\lambda \in \Lambda} A_\lambda$$

is a compact connected set. This collection being a chain means that for any λ_1 and λ_2, either $A_{\lambda_1} \subset A_{\lambda_2}$ or $A_{\lambda_2} \subset A_{\lambda_1}$.

Problem 30. Suppose that X is a Hausdorff space and that A and B are disjoint compact subsets of X. Show that there are disjoint sets U and V open in X such that $A \subset U$ and $B \subset V$.

Problem 31. Let $\mathbb{R}_s$ be the **Sorgenfrey line**. Show that $\mathbb{R}_s$ is a normal space.

Problem 32. Let $X \subset \mathbb{R}_s$. Show that X is normal in the subspace topology.

Problem 33. Show that a compact subset of $\mathbb{R}_s$ is countable.

Problem 34. Show that the rationals $\mathbb{Q}$ are dense in $\mathbb{R}_s$.

Problem 35. Show that $\mathbb{Q} \times \mathbb{Q} \subset \mathbb{R}_s \times \mathbb{R}_s$ is dense. Show that $\mathbb{R}_s \times \mathbb{R}_s$ has an uncountable closed discrete subspace.

Problem 36. State **Urysohn's Lemma** for normal spaces. Prove Urysohn's Lemma for **metric** spaces.

Problem 37. State and prove the **Tietze Extension Theorem** for normal spaces. Assume Urysohn's Lemma.

Problem 38. A topological space X is **second countable** provided that there is a countable base, $\mathscr{B} = \{U_i\}_{i=1}^{\infty}$, for the topology of X. Assume that X is a normal Hausdorff space that is second countable. Show that X is metrizable.

Problem 39. A space X is **first countable** provided that for each point $x \in X$, there is a countable set of neighborhoods of x, $\mathscr{B}_x = \{U_i\}_{i=1}^{\infty}$, such that for any open U with $x \in U$, there exists a $U_i \in \mathscr{B}_x$ with $x \in U_i \subset U$. Give an example of a first countable space X that has a countable dense set $A \subset X$ such that X is not metrizable.

Problem 40. Give an example of a space X which is $\mathbb{T}_1$ but not Hausdorff.

Problem 41. State and prove the **Contraction Mapping Theorem**.

Problem 42. State and prove the **Baire Category Theorem**.

Problem 43. Let X be a topological space. Define the **cone** of X, $c(X)$.

Problem 44. Show that if $X = \mathbb{N}$ with the discrete topology, then $c(\mathbb{N})$ is not metrizable.

Problem 45. Let X be a topological space and $f : X \to Y$ be a continuous function. Define the **mapping cylinder** of f denoted by M_f. Suppose that $Y = D^2$ is a disk and $f : \mathbb{S}^1 \to D^2$ is the constant function mapping $\mathbb{S}^1$ to the center point of the disk. Give a picture of the mapping cylinder.

Problem 46. Let X be a topological space. Define the **mapping torus** of X denoted by T_f. Let $X = \mathbb{S}^1$ and $f = id : \mathbb{S}^1 \to \mathbb{S}^1$. What is T_f in this case?

Problem 47. Suppose that $f : \mathbb{S}^1 \to \mathbb{S}^1$ is rotation by π. What is the mapping torus, T_f, in this case? What if f is rotation by $a \cdot \pi$ where a is an irrational number?

Problem 48. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and that $f([a, b]) \supset [c, d]$. Show that there is an interval $[h, j] \subset [a, b]$ such that $f([h, j]) = [c, d]$.

Problem 49. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and that $f([a, b]) \supset [a, b]$. Show that there is an $x \in [a, b]$ such that $f(x) = x$.

Problem 50. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and that f has a point of period three. Show that f has points of every prime period.

Problem 51. Show that there is a continuous function $f : [0, 1] \to [0, 1]^2$ which is onto.

Problem 52. Use the **Cantor Diagonal Process** to show that the Cantor set is not countable.

Problem 53. Let X be any set. Let $\mathscr{P}(X)$ be the set of all subsets of X. Show that there is no function $f : X \to \mathscr{P}(X)$ which is onto