L.E.J. BROUWER CHARACTERIZATION OF THE CANTOR SET

JAMES KEESLING

Let $C = \left\{\sum_{n=1}^{\infty} \frac{a_n}{3^n} \;\middle|\; a_n \in \{0,2\}\right\}$. This is the standard **Middle Third Cantor Set**. This space is non-empty, compact, metric, perfect, and has a countable base for the topology consisting of clopen sets. L.E.J. Brouwer showed that this is a topological characterization of this space in 1910 [1].

Theorem. *Let C and C' be any two topological spaces that are non-empty, compact, metric, perfect, and have a base for their topologies consisting of clopen sets. Then C ande C' are homeomorphic.*

.

Proof. First we note that if C satisfies the stated properties, then if $\mathscr{U}$ is any open cover of C, then there is a finite open cover $\mathscr{V} \prec \mathscr{U}$ such that $\mathscr{V}$ is a partition of C with all the sets in $\mathscr{V}$ non-empty. Let the cardinality of such a $\mathscr{V}$ be m. Then for any $n \geq m$, there is a $\mathscr{V}' \prec \mathscr{V}$ with the cardinality of $\mathscr{V}' = n$ and with all the sets in $\mathscr{V}'$ non-empty.

As a result, for C we have a sequence of covers by clopen sets which we denote $\mathscr{V}_n$ such that (1) the cardinality of $\mathscr{V}_n$ is m_n with all of the sets in $\mathscr{V}_n$ non-empty, (2) $\mathscr{V}_{n+1} \prec \mathscr{V}_n$ and hence $m_{n+1} \geq m_n$, (3) the diameter of each set $V \in \mathscr{V}_n$ less than $\frac{1}{n}$, (4) the clopen sets in $\mathscr{V}_n$ are pairwise disjoint, and (5) $C = \bigcap_{n=1}^{\infty} \bigcup \mathscr{V}_n$.

Now for C' we have a similar sequence with the same properties for C'. We will denote this sequence $\mathscr{V}'_n$. With some work we can also assume that $m_n = m'_n$ for each n. Furthermore, we can assume that there is a sequence of functions $F_n : \mathscr{V}_n \to \mathscr{V}'_n$ such that (1) F_n is one-to-one and onto. And, (2) if $V_1 \in \mathscr{V}_{n+1}$ and $V_2 \in \mathscr{V}_n$ with $V_1 \subset V_2$, then $F_{n+1}(V_1) \subset F_n(V_2)$ with $F_{n+1}(V_1) \in \mathscr{V}'_{n+1}$ and $F_n(V_2) \in \mathscr{V}'_n$.

This sequence of functions $F_n : \mathscr{V}_n \to \mathscr{V}'_n$ can be used to define a function $f : C \to C'$ in the following way. Let $x \in C$. Then for each n, there is a unique $V \in \mathscr{V}_n$ such that $x \in V$. Clearly, $x = \bigcap_{n=1}^{\infty} \left\{V \in \mathscr{V}_n \;\middle|\; x \in V\right\}$. Define $f(x) = \bigcap_{n=1}^{\infty} \left\{F_n(V) \in \mathscr{V}'_n \;\middle|\; x \in V\right\}$. By the above $F_n(V)$ is a nested sequence of non-empty sets in C' with diameters going to zero. Hence the intersection is a unique point and $f : C \to C'$ is well-defined.

For each $\mathscr{V}_n$ there is a Lebesgue number $\delta_n > 0$. Note that for each $\mathscr{V}'_n$, the diameter of each $V \in \mathscr{V}'_n$ is less than $\frac{1}{n}$. Now we will show that $f : C \to C'$ is uniformly continuous. Let $\epsilon > 0$ be given and let $\frac{1}{n_0} < \epsilon$. Let δ_{n_0} be the Lebesgue number for $\mathscr{V}_{n_0}$. Let x and y be in C with $d(x,y) < \delta_{n_0}$. Then $\{x,y\} \subset V_0$ for some $V_0 \in \mathscr{V}_{n_0}$ by the definition of the Lebesgue number. This implies that $\{f(x), f(y)\} \subset F_n(V_0) \in \mathscr{V}'_{n_0}$ and thus that $d(f(x), f(y)) < \frac{1}{n_0} < \epsilon$. Thus, f is uniformly continuous.

Our function $f : C \to C'$ is continuous, one-to-one, and onto. Thus, C and C' are homeomorphic. $\square$

As a result of this characterization, we say that any set C satisfying the properties is a **Cantor set**. Here are a few exercises to show the value of this characterization.

Exercise 1. Show that for each $0 < \epsilon < 1$ there is a Cantor set $C \subset [0,1]$ such that the Lebesgue measure of C is $\lambda(C) = \epsilon$.

Exercise 2. Show that C is homeomorphic to $C \times C$. Show that C is homeomorphic to C^∞.

Exercise 3. Suppose that X is a non-empty complete metric space with no isolated points. Then there is a Cantor set $C \subset X$. Note that this implies that $|X| \geq 2^{\aleph_0}$.

Exercise 4. Let X be any compact metric space. There is a continuous function $f : C \to X$ which is onto.

[1] L.E.J. Brouwer, *On the structure of perfect sets of points*, KNAW, Proceedings, **12** (1909-1910), 785-794 url