MTG 5317/4303 PROBLEMS

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These problems are based on the material covered in class. You should be able to provide proofs or examples for each problem. If you know these problems well, you should be prepared for the exams.

Problem 1. Suppose that X is a regular Lindelöf space. Show that X is normal.

Problem 2. Suppose that X is a second countable regular $\mathbf{T}_1$ space. Show that X is metrizable.

Problem 3. Suppose that X is complete metrizable, connected, and locally connected. Show that X is arcwise connected.

Problem 4. Suppose that X is compact metric, connected, and locally connected. Show that there is a continuous $f : [0, 1] \to X$ such that f is onto.

Problem 5. Suppose that X is compact and metrizable. Show that there is a continuous $f : C \to X$ which is onto where C is the Cantor set.

Problem 6. Suppose that $f, g : X \to \mathbb{R}^n$ are continuous functions. Show that f and g are homotopic.

Problem 7. Suppose that $f, g : X \to S^n$ are continuous maps such that for all $x \in X$, $f(x)$ and $g(x)$ are not antipodal. Show that f and g are homotopic.

Problem 8. Suppose that A is an absolute retract. Suppose that $f, g : X \to A$ are continuous functions. Show that f and g are homotopic.

Problem 9. Suppose that $X = \mathbb{R} \setminus \mathbb{Q}$ is the irrational numbers. Show that X is homeomorphic to $\prod_{i=1}^{\infty} \mathbb{Z}$.

Problem 10. Suppose that X is a complete separable metric space. Show that there is a continuous $f : \mathbb{R} \setminus \mathbb{Q} \to X$ which is onto.

Problem 11. Define an abstract simplicial complex and give a geometric realization of it. Define a simplicial map and give a geometric realization of the simplicial map.

Problem 12. Show that the Sorgenfrey line R is regular and Lindelöf. Show that $R \times R$ is not normal.

Problem 13. Prove the **Tietze Extension Theorem**. The theorem states: Suppose that X is normal and that $A \subset X$ is closed. Suppose that $f : A \to [0,1]$ is continuous. Then there is an $F : X \to [0,1]$ that is continuous such that $F|_A \equiv f$.

Problem 14. Prove the following using the Tietze Extension Theorem. Suppose that X is normal and that $A \subset X$ is closed. Suppose that $f : A \to [0,1]^n$ is continuous. Then there is an $F : X \to [0,1]^n$ that is continuous such that $F|_A \equiv f$.

Problem 15. Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed spaces. Suppose that $f_1, g_1 : (X, x_0) \to (Y, y_0)$ are homotopic and suppose that $f_2, g_2 : (Y, y_0) \to (Z, z_0)$ are homotopic. Show that $f_2 \circ f_1$ is homotopic to $g_2 \circ g_1$. We will denote this by $f_2 \circ f_1 \simeq \circ g_2 \circ g_1$.

Problem 16. Let (X, x_0) be a pointed space. A **loop** is a continuous function $f : [0,1] \to X$ such that $f(0) = f(1) = x_0$. The collection of loops is called the **loop space** and is denoted by $\Omega(X, x_0)$ or just by $\Omega(X)$. Define the **composition** of two loops f and g to be the loop $f \circ g$ where.

$$(f \circ g)(t) = \begin{cases} f(2 \cdot t) & : t \in \left[0, \frac{1}{2}\right] \\ g(2 \cdot t - 1) & : t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

Two loops $f, g \in \Omega(X, x_0)$ are **homotopic** provided that there is a homotopy $H : [0,1] \times [0,1] \to X$ such that

(1) $H(s, 0) \equiv f(s)$,
(2) $H(s, 1) \equiv g(s)$, and
(3) $H(0, t) \equiv H(1, t) \equiv x_0$.

Show the following. If $f_1, f_2 \in \Omega(X, x_0)$ and $g_1, g_2 \in \Omega(X, x_0)$ are homotopic loops, then $f_1 \circ g_1$ and $f_2 \circ g_2$ are homotopic loops.

Problem 17. Let (X, x_0) be a pointed space. The **trivial loop** is the function $f : [0,1] \to (X, x_0)$ such that $f(t) = x_0$ for all $t \in [0,1]$. Let $f : [0,1] \to (X, x_0)$ be a loop. Define f^{-1}

by $f^{-1}(t) = f(1-t)$. Show that f^{-1} is a loop. Show that $f \circ f^{-1}$ is homotopic to the trivial loop.

Problem 18. Consider three loops $f, g, h : [0,1] \to (X, x_0)$. Show that $(f \circ g) \circ h$ is homotopic to $f \circ (g \circ h)$.

Problem 19. Let (X, x_0) be a pointed space and let $\pi_1(X, x_0)$ be the collection of homotopy classes of loops. Show that $\pi_1(X, x_0)$ is a group under the following operation.

(1) Let $\alpha = [a]$ and $\beta = [b]$ be elements of $\pi_1(X, x_0)$. Define $\alpha \cdot \beta = [a \circ b]$. Show that $\alpha \cdot \beta$ is well-defined.
(2) Show that the operation $\alpha \cdot \beta$ is associative.
(3) Show that under the operation $\alpha \cdot \beta$, the homotopy class of the trivial loop is the identity.
(4) Show that under the operation $\alpha \cdot \beta$, the homotopy class of a^{-1} is the inverse of α.

Problem 20. Show that $\pi_1(S^1, 1) = \mathbb{Z}$.

Problem 21. Show that $\pi_1(S^n, 1) = 0$ for all $n > 1$.

Problem 22. Define a covering map and space for the pointed space (X, x_0).

Problem 23. Show that $e : \mathbb{R} \to S^1$ is a covering map where $e(t) = \exp(2\pi i \cdot t)$.

Problem 24. Show that $e^n : \mathbb{R}^n \to \mathbb{T}^n$ is a covering map where $e(t) = \exp(2\pi i \cdot t)$. Show that $\pi_1(\mathbb{T}^n) = \mathbb{Z}^n$ for all $n \geq 1$.

Problem 25. Suppose that $\pi_1(X, x_0) = G$ and that $\pi_1(Y, y_0) = H$. Show that $\pi_1(X, x_0) \times (Y, y_0) = G \times H$.

Problem 26. Let $P^n = S^n / \{-x, x\}$ for $n > 1$. Show that $\pi_1(P^n, x_0) = \mathbb{Z}_2$.

Problem 27. State the **Seifert-van Kampen Theorem**.

Problem 28. Use the Seifert-van Kampen Theorem to show that $\pi_1(\mathbb{T}^2) \cong \mathbb{Z} \times \mathbb{Z}$.

Problem 29. Let $k > 1$ be an integer. Let $S^1 = \partial D^2$ and $f : \partial D^2 \to S^1$ is defined by $f(z) = z^k$. Let X be the space obtained by attaching D^2 to S^1 by the map $f(z)$ on ∂D^2. Use the Seifert-van Kampen Theorem to show that $\pi_1(X) \cong \mathbb{Z}_k$.

Problem 30. Use the fundamental group to show that there is no retraction $r : D^2 \to \partial D^2$.

Problem 31. Show that if $f : D^2 \to D^2$ is a continuous map without a fixed point, then there is a retraction $r : D^2 \to \partial D^2$. Use this fact to prove the **Brouwer Fixed Point Theorem** for D^2.

Problem 32. The **Fundamental Theorem of Algebra** states that if $p(z) = a_n z^n + b_{n-1} z^{n-1} + \cdots + a_0$ is complex polynomial of degree $n > 0$, then $p(z)$ has a zero. Use the fundamental group to prove the Fundamental Theorem of Algebra.

Problem 33. Describe **Antoine's Necklace**. Let A denote Antoinne's Necklace in $\mathbb{R}^3$. Show that $\pi_1(\mathbb{R}^3 \setminus A) \neq 1$.

Problem 34. Describe the **Alexander Horned Sphere** as a subset of $\mathbb{R}^3$. Let S denote the Alexander Horned Sphere and O denote the unbounded component of its complement in $\mathbb{R}^3$. Show that $\pi_1(O) \neq 1$.

Problem 35. Assume that I^n has the fixed-point property. That is, assume the **Brouwer Fixed Point Theorem** for I^n. Show that any compact metric **Absolute Retract** has the fixed point property.

Problem 36. Consider the $n \times m$ matrix M with integer entries.

$$M = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1m} \\ k_{21} & k_{22} & \cdots & k_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ k_{n1} & k_{n2} & \cdots & k_{nm} \end{bmatrix}$$

This represents a group homomorphism $M : Z^m \to Z^n$. Show that there is a map $f : \mathbb{T}^m \to \mathbb{T}^n$ such that $f_* = M$ where $f_* : \pi_1(\mathbb{T}^m) \to \pi_1(\mathbb{T}^n)$ is the homomorphism induced by f on the fundamental group.

Problem 37. Let S_g be an orientable surface of genus g. Use the Seifert-van Kampen Theorem to show that $\pi_1(S_g) \cong F(a_1, b_1, a_2, b_2, \ldots, a_g, b_g)/ < a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} >$ where $F(x_1, x_2, \ldots, x_n)$ is the free group generated by n elements.

Problem 38. Let $G =< x_1, x_2, \ldots, x_n | R_1, R_2, \ldots, R_k >$ be a finitely presented group with n generators and k relations. Construct a compact metric space X such that $\pi_1(X) \cong G$.

Problem 39. Let $f : S^1 \to \mathbb{R}^2$ be a one-to-one continuous function. Show that $\mathbb{R}^2 \setminus f(S^1) = A \cup B$ where A and B are open and connected and $A \cap B = \emptyset$. This is a weak version of the **Jordan Curve Theorem**.