PROBLEMS FOR MTG 5317/4303 SPRING 2019

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1. Homotopy Theory

Problem 1. Let $f, g : X \to \mathbb{R}^n$ be continuous functions. Show that f and g are homotopic. We denote this by $f \sim g$.

Problem 2. Let $f, g : [0,1] \to \mathbb{R}^n$ be loops with base point 0. Show that f and g are homotopic as loops.

Problem 3. Let $\pi(t) = e^{2\pi i t} : \mathbb{R} \to \mathbb{S}^1$. Suppose that $\gamma : [0,1] \to \mathbb{S}^1$ is a loop with basepoint $1 \in \mathbb{S}^1$. Suppose that $\tilde{\gamma} : [0,1] \to \mathbb{R}$ such that $\pi \circ \tilde{\gamma} \equiv \gamma : [0,1] \to \mathbb{S}^1$ and $\tilde{\gamma}(0) = 0 \in \mathbb{R}$. Show that $\gamma(1) \in \mathbb{Z}$.

Problem 4. Let $\pi(t) = e^{2\pi i t} : \mathbb{R} \to \mathbb{S}^1$. Suppose that $\gamma : [0,1] \to \mathbb{S}^1$ is a loop with basepoint $1 \in \mathbb{S}^1$ and that γ' is also such a loop. Suppose that $\tilde{\gamma} : [0,1] \to \mathbb{R}$ such that $\pi \circ \tilde{\gamma} \equiv \gamma : [0,1] \to \mathbb{S}^1$ and $\tilde{\gamma}(0) = 0 \in \mathbb{R}$. Suppose that $\widetilde{\gamma'}$ has this property for γ'. Suppose that $\tilde{\gamma}(1) = \widetilde{\gamma'}(1)$. Show that γ and γ' are homotopic as loops.

Problem 5. Let $f, g : X \to \mathbb{S}^n$ are continuous such that for every $x \in X$, $f(x)$ and $g(x)$ are not antipodal. Show that f and g are homotopic.

Problem 6. Suppose that $n \geq 2$ and that $f : [0,1] \to \mathbb{S}^n$ is a loop with some basepoint $x_0 \in \mathbb{S}^n$. Show that f is homotopic to the constant loop $g : [0,1] \to \mathbb{S}^n$, $g(t) \equiv x_0$.

2. The Fundamental Group

Problem 7. Let $\alpha : [0,1] \to X$, $\beta : [0,1] \to X$, and $\gamma : [0,1] \to X$ be three loops in X the same basepoint x_0. Show that $(\alpha * \beta) * \gamma \neq \alpha * (\beta * \gamma)$. Show that $(\alpha * \beta) * \gamma \sim \alpha * (\beta * \gamma)$ as loops. We define $\alpha * \beta$ as the loop

$$\alpha * \beta = \begin{cases} \alpha(2t) & 0 \leq \frac{1}{2} \\ \beta(2t-1) & \frac{1}{2} \leq 1 \end{cases}$$

Problem 8. Let $\gamma : [0,1] \to X$ with basepoint x_0. Let γ^{-1} be defined by $\gamma^{-1}(t) = \gamma(1-t)$. Show that $\gamma * \gamma^{-1}$ is loop homotopic to the constant loop $1 : [0,1] \to X$, $1(t) \equiv x_0$.

Problem 9. Let (X, x_0) be a topological space X with basepoint x_0. This is called a **pointed topological space**. Let $\pi_1\left((X, x_0)\right) = \{[\gamma]\}$ be the loop homotopy classes of loops $\gamma : [0,1] \to (X, x_0)$ with operation $[\gamma] \cdot [\gamma'] = [\gamma * \gamma']$. Show that $\pi_1\left((X, x_0)\right)$ is a group.

Problem 10. Show that $\pi_1\left((\mathbb{S}^1, 1)\right) \cong \mathbb{Z}$.

Problem 11. Show that $\pi_1\left((X, x_0) \times (Y, y_0)\right) \cong \pi_1\left((X, x_0)\right) \times \pi_1\left((Y, y_0)\right)$.

Problem 12. Show that $\pi_1(\mathbb{T}^n) \cong \mathbb{Z}^n$.

Problem 13. Suppose that X and Y are topological spaces and that $f : X \to Y$ is continuous. Suppose that $f(x_0) = y_0$. Show that f induces a group homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ defined by $f_*([\gamma]) = [f \circ \gamma]$. Show that if $g : X \to Y$ is continuous with $g(x_0) = y_0$ with $f \sim g$ as pointed maps, then $f_* = g_*$.

Problem 14. Suppose that X, Y, and Z are topological spaces. Suppose that $f : X \to Y$ and $g : Y \to Z$ are continuous. Suppose that $f(x_0) = y_0$ and $g(y_0) = z_0$. Show that $(g \circ f)_* = g_* \circ f_*$ where the basepoints for X, Y, and Z are x_0, y_0, and z_0, respectively.

Problem 15. Suppose that $f : \mathbb{S}^1 \to \mathbb{S}^1$ is continuous. Show that f is homotopic to $z^n : \mathbb{S}^1 \to \mathbb{S}^1$ for some $n \in \mathbb{Z}$. This is the **degree** of f.

Problem 16. Let $f : \mathbb{S}^n \to \mathbb{S}^n$ and suppose that $f(x) = -f(-x)$ for all $x \in \mathbb{S}^n$. Then we say that f is an **antipodal map**. Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be an antipodal map. Show that f is homotopic to z^k for some odd integer k.

Problem 17. Let (X, x_0) and (Y, y_0) be pointed spaces. Define the **wedge** of (X, x_0) and (Y, y_0), $(X, x_0) \vee (Y, y_0)$, be the quotient space of the free union of X and Y identifying the points x_0 and y_0. Symbolically, $(X, x_0) \vee (Y, y_0) = (X \sqcup Y)/\{x_0 \sim y_0\}$. Use the Seifert-van Kampen Theorem to show that $\pi_1\left((\mathbb{S}^1, 1) \vee (\mathbb{S}^1, 1)\right) \cong \mathbb{Z} * \mathbb{Z}$. This group is the free group with two generators F_2. What group do you get with the wedge of three circles?

Problem 18. Show that $\pi_1(\mathbb{S}^2) = 1$ using the Seifert-van Kampen theorem.

Problem 19. Let $f : X \to X$ be a continuous map. The **Mapping Cone** of f is the quotient space C_f which is $X \times [0, 1]$ with $X \times \{1\}$ identified to a point and $(x, 0)$ identified with $f(x)$ for all $x \in X$. Consider $\mathbb{S}^1$ and let $f(z) = z^n : \mathbb{S}^1 \to \mathbb{S}^1$. Use the Seifert-van Kampen Theorem to show that $\pi_1(C_f) \cong \mathbb{Z}_n$.

3. The Brouwer Fixed Point Theorem and the Borsuk-Ulam Theorem

Problem 20. Let $f : I^2 \to I^2$ be a continuous function. Show that there is an $x \in I^2$ such that $f(x) = x$. This is called the **Brouwer Fixed Point Theorem** for I^2. The theorem is also true for I^n for all $n \geq 1$.

Problem 21. Let $f : \mathbb{S}^2 \to \mathbb{R}^2$ be a continuous function. Show that there is an $x \in \mathbb{S}^2$ such that $f(x) = f(-x)$. This is called the **Borsuk-Ulam Theorem** for $n = 2$. The theorem is also true for all $n \geq 1$.

4. Maps and Automorphisms of a Torus

Problem 22. Suppose that $f : \mathbb{T}^n \to \mathbb{T}^m$ is a continuous function taking $1 \in \mathbb{T}^n$ to $1 \in \mathbb{T}^m$. Show that $f_* : \pi_1(\mathbb{T}^n) \to \pi_1(\mathbb{T}^m)$ can be represented by a matrix M with integer entries such that multiplication by M represents the homomorphism $f_* = M : \mathbb{Z}^n \to \mathbb{Z}^m$. For each such matrix M with integer entries, find an $f : \mathbb{T}^n \to \mathbb{T}^m$ such that $f_* = M$.

Problem 23. Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be the **Arnold cat map**. By definition f is an automorphism of $\mathbb{T}^2$ such that $f_* : \pi_1(\mathbb{T}^2) \cong \mathbb{Z}^2 \to \pi_1(\mathbb{T}^2) \cong \mathbb{Z}^2$ is the matrix

$$f_* = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

What are the eigenvalues and eigenvectors of this matrix? How do these affect the map $f : \mathbb{T}^2 \to \mathbb{T}^2$?

Problem 24. Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be the Arnold cat map, Consider $\mathbb{T}^2$ as the quotient of $[0,1] \times [0,1]$ under the exponential map. Show that the periodic points of f are the image of the rational points under the quotient map.

Problem 25. Let $f : \mathbb{T}^n \to \mathbb{T}^n$ be such that f_* is given by the matrix

$$\begin{bmatrix} m_{11} & \cdots & m_{1n} \\ m_{21} & \cdots & m_{2,n} \\ \vdots & \ddots & \vdots \\ m_{n1} & \cdots & m_{nn} \end{bmatrix}.$$

What condition is necessary for f to be a homeomorphism? When the condition is met is there a homeomorphism g homotopic to f? What is the significance of $\det(M)$?

Problem 26. Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be the Arnold cat map. How would you find a point of period three? Of period one hundred one?

Problem 27. Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be an automorophism such that f_* is given by the matrix

$$f_* = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}.$$

Analyze the behavior this automorphism.

Problem 28. Let $f : \mathbb{T}^n \to \mathbb{T}^n$ be an automorophism such that f_* is given by the matrix

$$M = \begin{bmatrix} m_{11} & \cdots & m_{1n} \\ m_{21} & \cdots & m_{2,n} \\ \vdots & \ddots & \vdots \\ m_{n1} & \cdots & m_{nn} \end{bmatrix}.$$

We say that f is **hyperbolic** provided that for every eigenvalue λ of M $|\lambda| \neq 1$. What are the periodic points of f?

5. The Cantor Set

Problem 29. Suppose that C is the Cantor set. Show that C is homeomorphic to $C \times C$.

Problem 30. Suppose that C is the Cantor set. Show that there is a continuous $f : C \to [0,1]$ such that f is onto $[0,1]$ and such that for each $x \in [0,1]$, $f^{-1}(x)$ is homeomorphic to the Cantor set C.

Problem 31. Suppose that C is the Cantor set. Show that there is a continuous function $f : C \to [0,1] \times [0,1]$ which is onto. Show this using the fact that C is homeomorphic to $C \times C$.

Problem 32. Let X be a compact metric space. Use the Brouwer characterization of the Cantor set to show that there is a continuous function $f : C \to X$ which is onto.

6. Arcs and Maps of Arcs

Problem 33. Suppose that X is a compact metric space that is connected and locally connected. This is called a **Peano continuum**. Suppose that x and y are two distinct points in X. Show that there is a continuous $f : [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$ and such that f is one-to-one. This is called the **Arcwise Connectedness Theorem for Peano Continua**.

Problem 34. Suppose that X is a compact metric space that is connected and locally connected, i.e., X is a Peano continuum. Show that there is a continuous $f : [0,1] \to X$ which is onto. This is known as the **Hahn-Masurkiewicz Theorem**.

7. Lindelöf Spaces

Problem 35. A topological space is said to be **Lindelöf** provided that for every open cover $\mathscr{U}$ of X, there is a countable $\mathscr{V} \subset \mathscr{U}$ covering X. Show that if X is a separable metric space, then X is Lindelöf.

Problem 36. Suppose that a topological space X is regular and Lindelöf. Show that X is normal.

8. Compactifications

Problem 37. Let X be a topological space and C be a compact Hausdorff space such that there is an embedding $e : X \to C$ such that $e(X) \subset C$ is dense in C. The space C together with the embedding $e : X \to C$ is said to be a **compactification** for X. Suppose that X is a locally compact Hausdorff space. Show that there is a compactification of X obtained by adding one point to X. This is called the **one-point compactification** of X.

Problem 38. Let X be a Hausdorff space such that for each point $x \in X$ and each closed set $A \subset X$ such that $x \notin A$, there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f_A \equiv 1$. Such a space X is said to be **completely regular**. Show that for a completely regular space X, there is a compactification βX which has the property that for every continuous $f : X \to [0, 1]$, there is a continuous $\beta f : \beta X \to [0, 1]$ such that $\beta f_X \equiv f$. βX is called the **Stone-Čech compactification of** X.

Problem 39. State and prove the **Alexander Subbase Theorem**.

Problem 40. Show that if $\{X_\alpha\}_{\alpha \in A}$ is a collection of compact topological spaces, then $\prod_{\alpha \in A} X_\alpha$ is compact. This result is called the **Tychonoff Theorem**.

9. Surfaces

Problem 41. Let $n \geq 2$. Consider $\mathbb{S}^n/\{x, -x\} = \mathbb{P}^n$. Show that $\pi_1(\mathbb{P}^n) \cong \mathbb{Z}_2$.

Problem 42. Show that $\mathbb{P}^n$ is a compact $n-$manifold.

Problem 43. Let S be the $n-$fold connected sum of tori. Show that this space is a compact $2-$ manifold. Determine the fundamental group of this space.

Problem 44. Let S be the $n-$fold connected sum of tori together with the connected sum with $\mathbb{P}^2$. Show that this is a compact $2-$manifold. Determine the fundamental group of this space.

Problem 45. Take the connected sum of $\mathbb{P}^2$ with $\mathbb{P}^2$. What space do you get? Determine the fundamental group of this space.

Problem 46. Take the connected sum of $\mathbb{P}^2$ with $\mathbb{T}^2$. What space do you get? Determine the fundamental group of this space.

10. Absolute Retracts and Absolute Neighborhood Retracts

Problem 47. An **Absolute Retract (AR)** is a topological space X such that whenever $X \subset Y$ with Y a normal space and X closed in Y, then there is a retraction $r : Y \to X$. Show that $I = [0, 1]$ is an AR.

Problem 48. Show that if $\{X_i\}_{i=1}^{\infty}$ are all AR's, then $\prod_{i=1}^{\infty} X_i$ is also an AR. Show that if X is an AR and $A \subset X$ is a closed subset with a retraction $r : X \to A$, then A is also an AR.

Problem 49. Suppose that X is an AR. Then for any Z and any pair of continuous functions $f, g : Z \to X$, f and g are homotopic.

Problem 50. Define an ANR. Show that $\mathbb{S}^1$ is an ANR.

Problem 51. Suppose that X is an ANR and that $r : X \to A$ is a retraction onto a closed subset A of X. Show that A is also an ANR.

Problem 52. Suppose that X is a compact metric ANR. Show that there is an $\varepsilon > 0$ such that for any Z and any pair of continuous functions $f, g : Z \to X$ if $d_X(f(z), g(z)) < \varepsilon$ for all $x \in Z$, then f and g are homotopic.

11. The Jordan Curve Theorem

Problem 53. A **Jordan Curve** is an embedding of $\mathbb{S}^1$ into $\mathbb{R}^2$. State and prove the **Jordan Curve Theorem**.