Problem 1. Let \( f, g : X \to \mathbb{R}^n \) be continuous functions. Show that \( f \) and \( g \) are homotopic. We denote this by \( f \sim g \).

Problem 2. Let \( f, g : [0, 1] \to \mathbb{R}^n \) be loops with base point 0. Show that \( f \) and \( g \) are homotopic as loops.

Problem 3. Let \( \pi(t) = e^{2\pi it} : \mathbb{R} \to S^1 \). Suppose that \( \gamma : [0, 1] \to S^1 \) is a loop with basepoint \( 1 \in S^1 \). Suppose that \( \tilde{\gamma} : [0, 1] \to \mathbb{R} \) such that \( \pi \circ \tilde{\gamma} \equiv \gamma : [0, 1] \to S^1 \) and \( \tilde{\gamma}(0) = 0 \in \mathbb{R} \). Show that \( \gamma(1) \in \mathbb{Z} \).

Problem 4. Let \( \pi(t) = e^{2\pi it} : \mathbb{R} \to S^1 \). Suppose that \( \gamma : [0, 1] \to S^1 \) is a loop with basepoint \( 1 \in S^1 \) and that \( \gamma' \) is also such a loop. Suppose that \( \tilde{\gamma} : [0, 1] \to \mathbb{R} \) such that \( \pi \circ \tilde{\gamma} \equiv \gamma : [0, 1] \to S^1 \) and \( \tilde{\gamma}(0) = 0 \in \mathbb{R} \). Suppose that \( \gamma' \) has this property for \( \gamma' \). Suppose that \( \tilde{\gamma}(1) = \tilde{\gamma}'(1) \). Show that \( \gamma \) and \( \gamma' \) are homotopic as loops.

Problem 5. Suppose that \( C \) is the Cantor set. Show that \( C \) is homeomorphic to \( C \times C \).

Problem 6. Suppose that \( C \) is the Cantor set. Show that there is a continuous \( f : C \to [0, 1] \) such that \( f \) is onto \( [0, 1] \) and such that for each \( x \in [0, 1] \), \( f^{-1}(x) \) is homeomorphic to the Cantor set \( C \).

Problem 7. Suppose that \( C \) is the Cantor set. Show that there is a continuous function \( f : C \to [0, 1] \times [0, 1] \) which is onto. Show this using the fact that \( C \) is homeomorphic to \( C \times C \).

Problem 8. Let \( X \) be a compact metric space. Use the Brouwer characterization of the Cantor set to show that there is a continuous function \( f : C \to X \) which is onto.

Problem 9. Let \( f, g : X \to S^n \) be continuous such that for every \( x \in X \), \( f(x) \) and \( g(x) \) are not antipodal. Show that \( f \) and \( g \) are homotopic.
Problem 10. Suppose that \( n \geq 2 \) and that \( f : [0,1] \to S^n \) is a loop with some basepoint \( x_0 \in S^n \). Show that \( f \) is homotopic to the constant loop \( g : [0,1] \to S^n \), \( g(t) \equiv x_0 \).

Problem 11. Let \( \alpha : [0,1] \to X \), \( \beta : [0,1] \to X \), and \( \gamma : [0,1] \to X \) be three loops in \( X \) the same basepoint \( x_0 \). Show that \((\alpha \ast \beta) \ast \gamma \neq \alpha \ast (\beta \ast \gamma)\). Show that \((\alpha \ast \beta) \ast \gamma \sim \alpha \ast (\beta \ast \gamma)\) as loops. We define \( \alpha \ast \beta \) as the loop

\[
\alpha \ast \beta = \begin{cases} 
\alpha(2t) & 0 \leq \frac{t}{2} \\
\beta(2t - 1) & \frac{t}{2} \leq 1
\end{cases}
\]

Problem 12. Let \( \gamma : [0,1] \to X \) with basepoint \( x_0 \). Let \( \gamma^{-1} \) be defined by \( \gamma^{-1}(t) = \gamma(1-t) \). Show that \( \gamma \ast \gamma^{-1} \) is loop homotopic to the constant loop \( 1 : [0,1] \to X \), \( 1(t) \equiv x_0 \).

Problem 13. Let \((X,x_0)\) be a topological space \( X \) with basepoint \( x_0 \). This is called a pointed topological space. Let \( \pi_1((X,x_0)) = \{[\gamma]\} \) be the loop homotopy classes of loops \( \gamma : [0,1] \to (X,x_0) \) with operation \([\gamma] \cdot [\gamma'] = [\gamma \ast \gamma']\). Show that \( \pi_1((X,x_0)) \) is a group.

Problem 14. Show that \( \pi_1((S^1,1)) \cong \mathbb{Z} \).

Problem 15. Show that \( \pi_1((X,x_0) \times (Y,y_0)) \cong \pi_1((X,x_0)) \times \pi_1((Y,y_0)) \).

Problem 16. Show that \( \pi_1(\mathbb{T}^n) \cong \mathbb{Z}^n \).

Problem 17. Suppose that \( X \) and \( Y \) are topological spaces and that \( f : X \to Y \) is continuous. Suppose that \( f(x_0) = y_0 \). Show that \( f \) induces a group homomorphism \( f_* : \pi_1(X,x_0) \to \pi_1(Y,y_0) \) defined by \( f_*([\gamma]) = [f \circ \gamma] \). Show that if \( g : X \to Y \) is continuous with \( g(x_0) = y_0 \) with \( f \sim g \) as pointed maps, then \( f_* = g_* \).

Problem 18. Suppose that \( X, Y, \) and \( Z \) are topological spaces. Suppose that \( f : X \to Y \) and \( g : Y \to Z \) are continuous. Suppose that \( f(x_0) = y_0 \) and \( g(y_0) = z_0 \). Show that \((g \circ f)_* = g_* \circ f_* \) where the basepoints for \( X, Y, \) and \( Z \) are \( x_0, y_0, \) and \( z_0 \), respectively.

Problem 19. Let \( f : I^2 \to I^2 \) be a continuous function. Show that there is an \( x \in I^2 \) such that \( f(x) = x \). This is called the Brouwer Fixed Point Theorem for \( I^2 \). The theorem is also true for \( I^n \) for all \( n \geq 1 \).
Problem 20. Suppose that $f : S^1 \to S^1$ is continuous. Show that $f$ is homotopic to $z^n : S^1 \to S^1$ for some $n \in \mathbb{Z}$. This is the degree of $f$.

Problem 21. Let $f : S^n \to S^n$ and suppose that $f(x) = -f(-x)$ for all $x \in S^n$. Then we say that $f$ is an antipodal map. Let $f : S^1 \to S^1$ be an antipodal map. Show that $f$ is homotopic to $z^k$ for some odd integer $k$.

Problem 22. Let $f : S^2 \to \mathbb{R}^2$ be a continuous function. Show that there is an $x \in S^2$ such that $f(x) = f(-x)$. This is called the Borsuk-Ulam Theorem for $n = 2$. The theorem is also true for all $n \geq 1$.

Problem 23. Let $(X,x_0)$ and $(Y,y_0)$ be pointed spaces. Define the wedge of $(X,x_0)$ and $(Y,y_0)$, $(X,x_0) \vee (Y,y_0)$, be the quotient space of the free union of $X$ and $Y$ identifying the points $x_0$ and $y_0$. Symbolically, $(X,x_0) \vee (Y,y_0) = (X \sqcup Y)/\{x_0 \sim y_0\}$. Use the Seifert-van Kampen Theorem to show that $\pi_1((S^1, 1) \vee (S^1, 1)) \cong \mathbb{Z} * \mathbb{Z}$. This group is the free group with two generators $F_2$. What group do you get with the wedge of three circles?

Problem 24. Let $f : X \to X$ be a continuous map. The Mapping Cone of $f$ is the quotient space $C_f$ which is $X \times [0, 1]$ with $X \times \{1\}$ identified to a point and $(x, 0)$ identified with $f(x)$ for all $x \in X$. Consider $S^1$ and let $f(z) = z^n : S^1 \to S^1$. Use the Seifert-van Kampen Theorem to show that $\pi_1(C_f) \cong \mathbb{Z}_n$.

Problem 25. Suppose that $f : \mathbb{T}^n \to \mathbb{T}^m$ is a continuous function taking $1 \in \mathbb{T}^n$ to $1 \in \mathbb{T}^m$. Show that $f_* : \pi_1(\mathbb{T}^n) \to \pi_1(\mathbb{T}^m)$ can be represented by a matrix $M$ with integer entries such that multiplication by $M$ represents the homomorphism $f_* = M : \mathbb{Z}^n \to \mathbb{Z}^m$. For each such matrix $M$ with integer entries, find an $f : \mathbb{T}^n \to \mathbb{T}^m$ such that $f_* = M$.

Problem 26. Suppose that $X$ is a compact metric space that is connected and locally connected. This is called a Peano continuum. Suppose that $x$ and $y$ are two distinct points in $X$. Show that there is a continuous $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$ and such that $f$ is one-to-one. This is called the Arcwise Connectedness Theorem for Peano Continua.

Problem 27. Suppose that $X$ is a compact metric space that is connected and locally connected, i.e., $X$ is a Peano continuum. Show that there is a continuous $f : [0, 1] \to X$ which is onto. This is known as the Hahn-Masurkiewicz Theorem.
Problem 28. A topological space is said to be Lindelöf provided that for every open cover $\mathcal{U}$ of $X$, there is a countable $\mathcal{V} \subset \mathcal{U}$ covering $X$. Show that if $X$ is a separable metric space, then $X$ is Lindelöf.

Problem 29. Suppose that a topological space $X$ is regular and Lindelöf. Show that $X$ is normal.