1. ARCS AND MAPS OF ARCS

**Problem 33.** Suppose that \( X \) is a compact metric space that is connected and locally connected. This is called a **Peano continuum**. Suppose that \( x \) and \( y \) are two distinct points in \( X \). Show that there is a continuous \( f : [0,1] \to X \) such that \( f(0) = x \) and \( f(1) = y \) and such that \( f \) is one-to-one. This is called the **Arcwise Connectedness Theorem for Peano Continua**.

**Problem 34.** Suppose that \( X \) is a compact metric space that is connected and locally connected, i.e., \( X \) is a Peano continuum. Show that there is a continuous \( f : [0,1] \to X \) which is onto. This is known as the **Hahn-Masurkiewicz Theorem**.

2. LINDELÖF SPACES

**Problem 35.** A topological space is said to be **Lindelöf** provided that for every open cover \( \mathcal{U} \) of \( X \), there is a countable \( \mathcal{V} \subset \mathcal{U} \) covering \( X \). Show that if \( X \) is a separable metric space, then \( X \) is Lindelöf.

**Problem 36.** Suppose that a topological space \( X \) is regular and Lindelöf. Show that \( X \) is normal.

3. COMPACTIFICATIONS

**Problem 37.** Let \( X \) be a topological space and \( C \) be a compact Hausdorff space such that there is an embedding \( e : X \to C \) such that \( e(X) \subset C \) is dense in \( C \). The space \( C \) together with the embedding \( e : X \to C \) is said to be a **compactification** for \( X \). Suppose that \( X \) is a locally compact Hausdorff space. Show that there is a compactification of \( X \) obtained by adding one point to \( X \). This is called the **one-point compactification** of \( X \).

**Problem 38.** Let \( X \) be a Hausdorff space such that for each point \( x \in X \) and each closed set \( A \subset X \) such that \( x \notin A \), there is a continuous function \( f : X \to [0,1] \) such that \( f(x) = 0 \) and \( f_A \equiv 1 \). Such a space \( X \) is said to be **completely regular**. Show that for a completely regular space \( X \), there is a compactification \( \beta X \) which has the property
that for every continuous $f : X \rightarrow [0, 1]$, there is a continuous $\beta f : \beta X \rightarrow [0, 1]$ such that $\beta f_X \equiv f$. $\beta X$ is called the \textbf{Stone-Čech compactification of $X$}.

**Problem 39.** State and prove the \textbf{Alexander Subbase Theorem}.

**Problem 40.** Show that if $\{X_\alpha\}_{\alpha \in A}$ is a collection of compact topological spaces, then $\prod_{\alpha \in A} X_\alpha$ is compact. This result is called the \textbf{Tychonoff Theorem}.

4. \textbf{Surfaces}

**Problem 41.** Let $n \geq 2$. Consider $S^n / \{x, -x\} = \mathbb{P}^n$. Show that $\pi_1(\mathbb{P}^n) \cong \mathbb{Z}_2$.

**Problem 42.** Show that $\mathbb{P}^n$ is a compact $n$–manifold.

**Problem 43.** Let $S$ be the $n$–fold connected sum of tori. Show that this space is a compact $2$–manifold. Determine the fundamental group of this space.

**Problem 44.** Let $S$ be the $n$–fold connected sum of tori together with the connected sum with $\mathbb{P}^2$. Show that this is a compact $2$–manifold. Determine the fundamental group of this space.

**Problem 45.** Take the connected sum of $\mathbb{P}^2$ with $\mathbb{P}^2$. What space do you get? Determine the fundamental group of this space.

**Problem 46.** Take the connected sum of $\mathbb{P}^2$ with $T^2$. What space do you get? Determine the fundamental group of this space.
5. **Absolute Retracts and Absolute Neighborhood Retracts**

**Problem 47.** An **Absolute Retract (AR)** is a topological space $X$ such that whenever $X \subset Y$ with $Y$ a normal space and $X$ closed in $Y$, then there is a retraction $r : Y \to X$. Show that $I = [0, 1]$ is an AR.

**Problem 48.** Show that if $\{X_i\}_{i=1}^{\infty}$ are all AR’s, then $\prod_{i=1}^{\infty} X_i$ is also an AR. Show that if $X$ is an AR and $A \subset X$ is a closed subset with a retraction $r : X \to A$, then $A$ is also an AR.

**Problem 49.** Suppose that $X$ is an AR. Then for any $Z$ and any pair of continuous functions $f, g : Z \to X$, $f$ and $g$ are homotopic.

**Problem 50.** Define an ANR. Show that $S^1$ is an ANR.

**Problem 51.** Suppose that $X$ is an ANR and that $r : X \to A$ is a retraction onto a closed subset $A$ of $X$. Show that $A$ is also an ANR.

**Problem 52.** Suppose that $X$ is a compact metric ANR. Show that there is an $\varepsilon > 0$ such that for any $Z$ and any pair of continuous functions $f, g : Z \to X$ if $d_X(f(z), g(z)) < \varepsilon$ for all $x \in Z$, then $f$ and $g$ are homotopic.