QUIZ 6 MTG 5317/4303 SPRING 2019

JAMES KEESLING

1. Arcs and Maps of Arcs

Problem 33. Suppose that X is a compact metric space that is connected and locally connected. This is called a **Peano continuum**. Suppose that x and y are two distinct points in X. Show that there is a continuous $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$ and such that f is one-to-one. This is called the **Arcwise Connectedness Theorem for Peano Continua**.

Problem 34. Suppose that X is a compact metric space that is connected and locally connected, i.e., X is a Peano continuum. Show that there is a continuous $f : [0, 1] \to X$ which is onto. This is known as the **Hahn-Masurkiewicz Theorem**.

2. Lindelöf Spaces

Problem 35. A topological space is said to be **Lindelöf** provided that for every open cover $\mathscr{U}$ of X, there is a countable $\mathscr{V} \subset \mathscr{U}$ covering X. Show that if X is a separable metric space, then X is Lindelöf.

Problem 36. Suppose that a topological space X is regular and Lindelöf. Show that X is normal.

3. Compactifications

Problem 37. Let X be a topological space and C be a compact Hausdorff space such that there is an embedding $e : X \to C$ such that $e(X) \subset C$ is dense in C. The space C together with the embedding $e : X \to C$ is said to be a **compactification** for X. Suppose that X is a locally compact Hausdorff space. Show that there is a compactification of X obtained by adding one point to X. This is called the **one-point compactification** of X.

Problem 38. Let X be a Hausdorff space such that for each point $x \in X$ and each closed set $A \subset X$ such that $x \notin A$, there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f_A \equiv 1$. Such a space X is said to be **completely regular**. Show that for a completely regular space X, there is a compactification βX which has the property that for every continuous $f : X \to [0,1]$, there is a continuous $\beta f : \beta X \to [0,1]$ such that $\beta f_X \equiv f$. βX is called the **Stone-Čech compactification of** X.

Problem 39. State and prove the **Alexander Subbase Theorem**.

Problem 40. Show that if $\{X_\alpha\}_{\alpha\in A}$ is a collection of compact topological spaces, then $\prod_{\alpha\in A} X_\alpha$ is compact. This result is called the **Tychonoff Theorem**.

4. Surfaces

Problem 41. Let $n \geq 2$. Consider $\mathbb{S}^n/\{x,-x\} = \mathbb{P}^n$. Show that $\pi_1(\mathbb{P}^n) \cong \mathbb{Z}_2$.

Problem 42. Show that $\mathbb{P}^n$ is a compact $n-$manifold.

Problem 43. Let S be the $n-$fold connected sum of tori. Show that this space is a compact $2-$ manifold. Determine the fundamental group of this space.

Problem 44. Let S be the $n-$fold connected sum of tori together with the connected sum with $\mathbb{P}^2$. Show that this is a compact $2-$manifold. Determine the fundamental group of this space.

Problem 45. Take the connected sum of $\mathbb{P}^2$ with $\mathbb{P}^2$. What space do you get? Determine the fundamental group of this space.

Problem 46. Take the connected sum of $\mathbb{P}^2$ with $\mathbb{T}^2$. What space do you get? Determine the fundamental group of this space.

5. Absolute Retracts and Absolute Neighborhood Retracts

Problem 47. An **Absolute Retract (AR)** is a topological space X such that whenever $X \subset Y$ with Y a normal space and X closed in Y, then there is a retraction $r : Y \to X$. Show that $I = [0, 1]$ is an AR.

Problem 48. Show that if $\{X_i\}_{i=1}^{\infty}$ are all AR's, then $\prod_{i=1}^{\infty} X_i$ is also an AR. Show that if X is an AR and $A \subset X$ is a closed subset with a retraction $r : X \to A$, then A is also an AR.

Problem 49. Suppose that X is an AR. Then for any Z and any pair of continuous functions $f, g : Z \to X$, f and g are homotopic.

Problem 50. Define an ANR. Show that $\mathbb{S}^1$ is an ANR.

Problem 51. Suppose that X is an ANR and that $r : X \to A$ is a retraction onto a closed subset A of X. Show that A is also an ANR.

Problem 52. Suppose that X is a compact metric ANR. Show that there is an $\varepsilon > 0$ such that for any Z and any pair of continuous functions $f, g : Z \to X$ if $d_X(f(z), g(z)) < \varepsilon$ for all $x \in Z$, then f and g are homotopic.