Newton Iteration

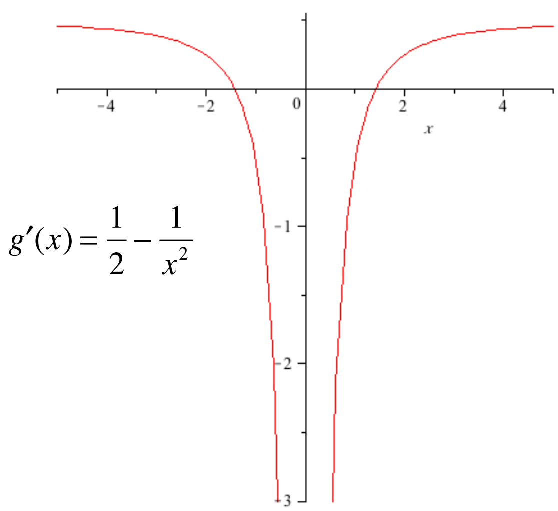
Consider solution to the equation $z^2 = 2$ on the real line. The solution is clearly $z = \pm\sqrt{2}$. Now consider approximating the solutions of this equation using Newton's Method. We first set up the function to be set to zero, $f(x) = x^2 - 2 = 0$. Then our Newton Function is the following.

$$g(x) = x - \frac{x^2 - 2}{2x}$$

The derivative of $g(x)$ is the following.

$$g'(x) = \frac{1}{2} - \frac{1}{x^2}$$

Clearly $g'(x) = 0$ if and only if $x = \pm\sqrt{2}$. The graph of $g'(x)$ is given below.



Based on the graph of $g'(x)$ and using the Mean Value Theorem, we can analyze exactly what will happen when we apply the Newton Function to any starting value. If we have any positive number $0 < x < \sqrt{2}$, then the Mean Value Theorem says that $g(x) > \sqrt{2}$. For $x > \sqrt{2}$, we have that $g(x) > \sqrt{2}$ as well. We also have that $\left|g(x) - \sqrt{2}\right| < \frac{1}{2}\left|x - \sqrt{2}\right|$.

So, for $x > \sqrt{2}$ we have that $\left|g^n(x) - \sqrt{2}\right| < \left(\frac{1}{2}\right)^n \left|x - \sqrt{2}\right|$. This guarantees that $g^n(x) \to \sqrt{2}$ as $n \to \infty$ whether x is less than $\sqrt{2}$ or greater than $\sqrt{2}$ as long as x is positive. If x is negative, then $g^n(x) \to -\sqrt{2}$ by a similar argument. So, Newton's

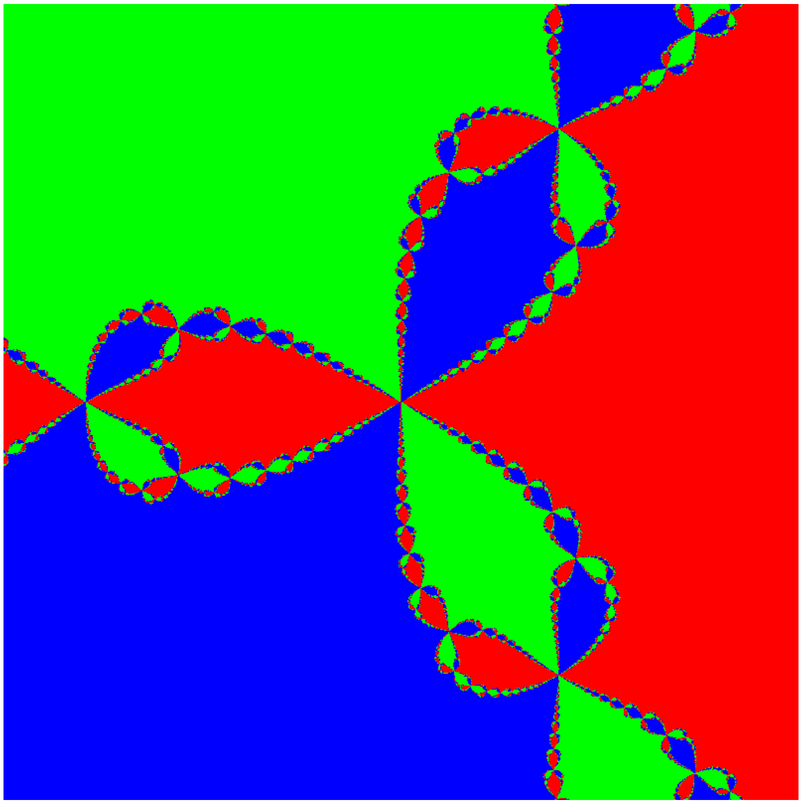
Method converges for all x except $x = 0$. On the other hand, we get very complicated patterns of convergence when we consider the cube roots of unity in the complex plane. We also get a complicated set of points where the Newton function does not converge.

Below is a graph of the convergence in the complex plane of Newton's Method for the following equation.

$$f(z) = z^3 - 1 = 0$$

The Newton Function is $g(z) = z - \dfrac{z^3 - 1}{3z^2}$.

There are three roots to the equation, $z = 1, \sqrt[3]{-1}, \left(\sqrt[3]{-1}\right)^2$. The points in the plane are color coded by which points converge to these respective points. You can see the interesting pattern that is made. The complement of the points that converge is a fractal set and is non-empty. It is an example of a *Julia Set*.



The Newton Method for approximating the roots of an equation is several hundred years old. The discovery about the intricate pattern made by the points that do not converge is very recent. There are new discoveries still being made about this old algorithm. Mathematics is not a static field that never changes. It is fresh and vibrant. This is just one example.