Closed Newton-Cotes Integration

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This document will discuss Newton-Cotes Integration. Other methods of numerical integration will be discussed in other posts. The other methods will include the Trapezoidal Rule, Romberg Integration, and Gaussian Integration.

1 Principle of Newton-Cotes Integration

We only cover the Newton-Cotes closed formulas. The interval of integration $[a, b]$ is partitioned by the points.

$$\left\{a, a + \frac{b-a}{n}, a + 2 \cdot \frac{b-a}{n}, \ldots, b\right\}$$

We estimate the integral of $f(x)$ on this interval by using the Lagrange interpolating polynomial through the following points.

$$\left\{(a, f(a)), \left(a + \frac{b-a}{n}, f\left(a + \frac{b-a}{n}\right)\right), \ldots, (b, f(b))\right\}$$

The formula for the integral of this Lagrange polynomial simplifies to a linear combination of the values of $f(x)$ at the points

$$\left\{x_i = a + i \cdot \frac{b-a}{n} \,\middle|\, i = 0, 1, 2, \ldots, n\right\}.$$

In the next section we give a method for calculating the coefficients for this linear combination.

2 The Newton-Cotes Closed Formula

We wish to estimate the following integral.

$$\int_a^b f(x)dx$$

We use the value of the function at the following points $\{a + i \cdot \frac{b-a}{n} \mid i = 0, 1, 2, \ldots, n\}$. Our estimate will have the following form.

$$\int_a^b f(x)dx = A_0 \cdot f(a) + A_1 \cdot f\left(a + \frac{b-a}{n}\right) + A_2 \cdot f\left(a + 2 \cdot \frac{b-a}{n}\right) + \cdots + A_n \cdot f(b)$$

So, what values should we use for the coefficients and how can we calculate them? There are several approaches to this. It turns out that these $\{A_0, A_1, \ldots, A_n\}$ coefficients will be proportional to the length of the interval, $b - a$. We use the interval $[0, n]$ and the points $\{0, 1, 2, \ldots, n\}$ and normalize the coefficients we get by dividing by the length of the interval, n. By this means we get a normalized set of coefficients $\{a_0, a_1, \ldots, a_n\}$. We would then have

$$A_i = (b - a) \cdot a_i$$

as our coefficients for a particular interval $[a, b]$. Our estimate of the integral will then be given by the following.

$$\int_a^b f(x)\,dx \approx (b - a) \cdot \sum_{i=0}^{n} a_i \cdot f\left(a + i \cdot \frac{b-a}{n}\right)$$

We now compute the normalized coefficients $\{a_0, a_1, \ldots, a_n\}$. Let M be an $(n + 1) \times (n + 1)$ Vandermonde matrix.

$$M = \text{Vandermonde}([0, 1, 2, \ldots, n]) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2^2 & \cdots & 2^n \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & n & n^2 & \cdots & n^n \end{bmatrix}$$

Let M^T be the transpose of M. Let A be a column vector with entries $n \cdot a_i$. Let B be a column vector with entries

$$b_i = \int_0^n x^i dx = \frac{n^{i+1}}{i+1}.$$

Then we get the matrix equation

$$M^T \cdot A = B.$$

Solving for A we get the following.

$$A = \left(M^T\right)^{-1} \cdot B$$

Our normalized coefficients are $\frac{1}{n} \cdot A$.

Example 2.1. *Let* $n = 5$. *Determine* $\{a_0, a_1, a_2, a_3, a_4, a_5\}$.

The Vandermonde matrix that was used in the above analysis is the following for $n = 5$.

$$\text{Vandermonde}([0,1,2,3,4,5]) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 & 32 \\ 1 & 3 & 9 & 27 & 81 & 243 \\ 1 & 4 & 16 & 64 & 256 & 1024 \\ 1 & 5 & 25 & 125 & 625 & 3125 \end{bmatrix}$$

The normalized coefficients are the following.

$$\left\{\frac{19}{288}, \frac{25}{96}, \frac{25}{144}, \frac{25}{144}, \frac{25}{96}, \frac{19}{288}\right\}$$

3 Alternative Method for Determining the Normalized Coefficients

As stated at the beginning of this section, the Newton-Cotes estimate uses the integral of the Lagrange interpolating polynomial through the following points.

$$\{(a, f(a)), \left(a + \frac{b-a}{n}, f\left(a + \frac{b-a}{n}\right)\right), \ldots, (b, f(b))\}$$

Consider the points $\left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\right\}$. Then the normalized coefficient a_i is given by the following integral formula.

$$a_i = \int_0^1 L_i(x)dx$$

You can easily verify the numbers in Example 2.1 using the program for the Lagrange interpolating polynomial and integrating.

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lagrange([0,1/5,2/5,3/5,4/5,1], [0,0,1,0,0,0])
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The polynomial p given by the program is the Lagrange polynomial that is one at $\frac{2}{5}$ and zero at the other points. The coefficient a_2 is given by integrating p. We get the correct number.

$$a_2 = \int_0^1 p\,dx = \frac{25}{144}$$

4 TN-Inspire CX CAS Program for Closed Newton-Cotes

Here is a TN-Inspire CX CAS program for the closed Newton-Cotes normalized coefficients. The input variable is the number n in the above analysis. The output is the column vector "coef" whose components are the $n+1$ normalized coefficients.

Figure 1: Screenshot of the program Newton-Cotes closed coefficients with an example

5 Program for the Newton-Cotes Integral

Here is a program that computes the Newton-Cotes estimate of the integral $\int_a^b f(x)\,dx$ using $n+1$ equally spaced points in the interval $[a,b]$. The variable f is the given function with x as the assumed variable. The variables a and b are the endpoints of the interval of integration. The variable n is the number of intervals into which $[a,b]$ is subdivided. The output variable *ncot* is the estimate of the integral.

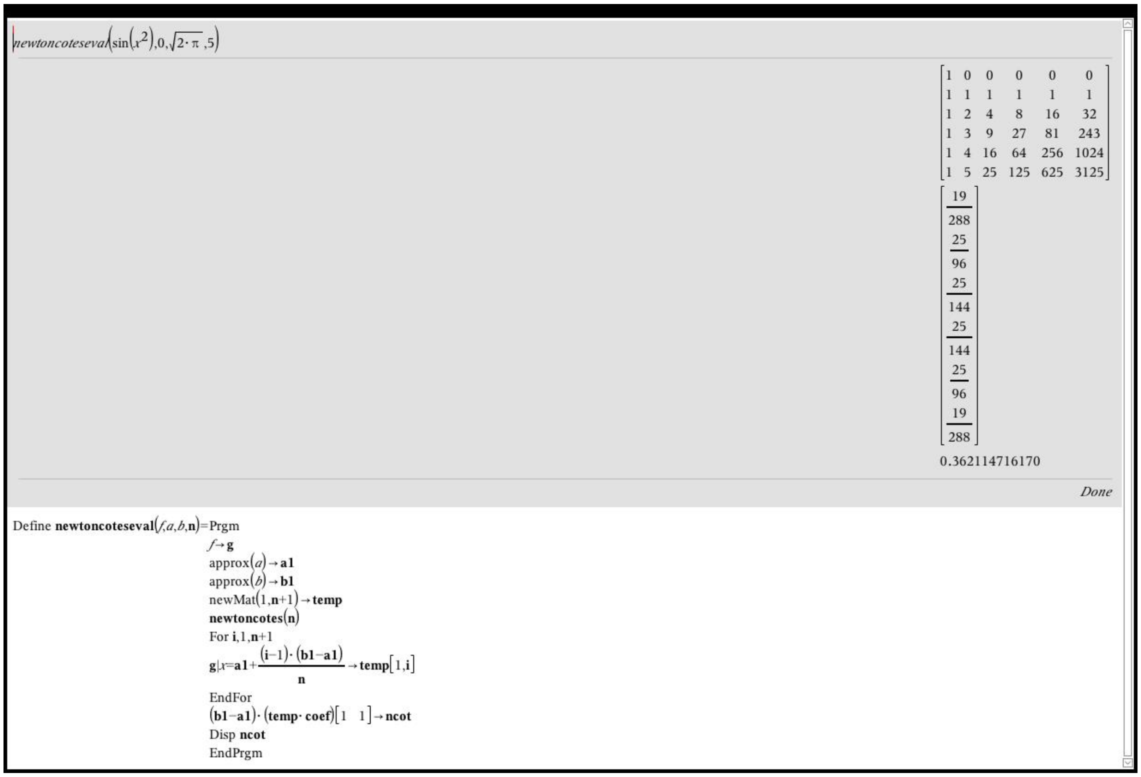

Figure 2: Screenshot of the program Newton-Cotes evaluation of the integral

6 Error Analysis of Newton-Cotes

One would expect that the error would grow smaller as we use larger n in the Newton-Cotes method. It turns out that this is not correct. The reason is that using equally spaced points, the Lagrange interpolating polynomial may give a very bad approximation of the function away from the interpolation points. It can be so bad that the integrals of these polynomials do not converge to the integral of the function $f(x)$ as $n \to \infty$. The classic example is the following function over the interval $[-4, 4]$.

$$f(x) = \frac{1}{1+x^2}$$

Here is an example of how divergent the Lagrange polynomial can be in this case. The graph plots $\frac{1}{1+x^2}$ and the Lagrange polynomial for twenty-one equally spaced points on the interval $[-4, 4]$.

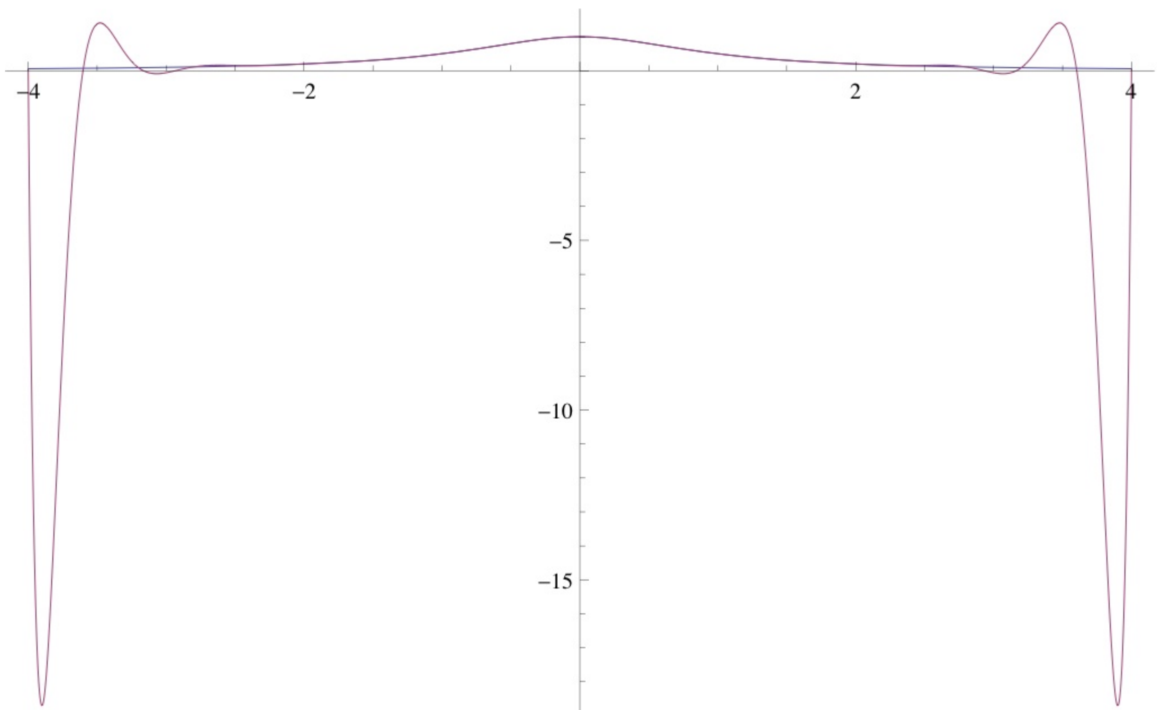

Figure 3: Plot of $\frac{1}{1+x^2}$ and Diverging Lagrange Polynomial