THE NEWTON-RAPHSON METHOD

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Consider the equation $f(x) = 0$ where f is a continuously differentiable real-valued function of a real variable. Suppose that z is a point where $f(z) = 0$, a solution of our original problem. Newton's idea was to start with an estimate x_0 near z and improve that estimate by taking the tangent line over x_0 and letting x_1 be the point where the tangent line crosses the $x-$axis. If the point x_0 is close enough to z, the estimate x_1 will be closer to z than x_0. Graphically we have the following situation. We will explain.

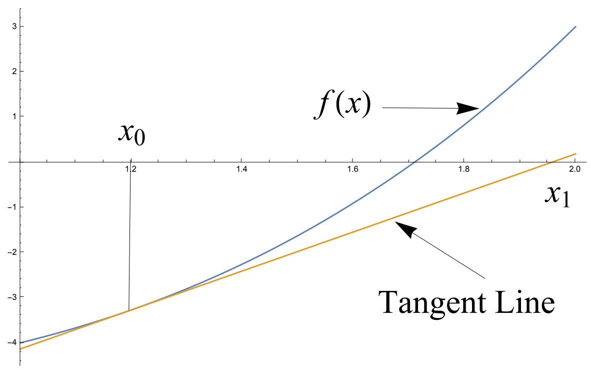

FIGURE 1. Graph of a function showing how x_1 depends on x_0

1. Theory Behind the Method

The formula for x_1 in terms of x_0 is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

We define the *Newton Function* to be $g(x) = x - \frac{f(x)}{f'(x)}$. This is the formula for x_1 in terms of x_0, but ignoring the fact that x_0 may have been chosen in some special way. We observe some properties of this function.

The first note is that $f(z) = 0$ if and only if $g(z) = z$ assuming that $f'(z) \neq 0$. We have converted the problem from finding a **zero** for the function $f(x)$ to finding a **fixed point** for the related function $g(x)$.

We use the **Mean Value Theorem** to show that if x_0 is close enough to z, then $f(x_0) = x_1$ is closer to z than x_0.

Theorem 1. *Suppose that h is continuous on the interval $[a, b]$ and differentiable on (a, b). Then there is a c, $a < c < b$ such that*

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}$$

This is a theorem from Calculus I. We will assume such results for this class.

Note that

$$g'(x) = \frac{f(x) \cdot f''(x)}{(f'(x))^2}.$$

Thus, we have $g'(z) = 0$. Let $h > 0$ be such that for all $x \in [z - h, z + h]$, $|g(x)'| < \frac{1}{2}$. What we claim is that for any $x_0 \in (z - h, z + h)$, $g(x_0)$ is closer to z than x_0. We now show this using the Mean Value Theorem. Consider the Mean Value Theorem applied to the points z and x_0. There must be a c between z and x_0 such that

$$g'(c) = \frac{g(x_0) - g(z)}{x_0 - z} = \frac{g(x_0) - z}{x_0 - z}$$

$$\frac{1}{2} > |g'(c)| = \left| \frac{g(x_0) - z}{x_0 - z} \right| = \frac{|g(x_0) - z|}{|x_0 - z|}.$$

So, we have that $|g(x_0) - z| < \frac{1}{2}|x_0 - z|$. In particular $x_1 \in [z - h, z + h]$. So, by induction we have that

$$|g^n(x_0) - z| < \left(\frac{1}{2}\right)^n |x_0 - z|.$$

So, we have that $g^n(x_0) \to z$ as $n \to \infty$.

One more theorem is useful here.

Theorem 2. *Suppose that we have any continuous function $h(x)$ and suppose that x_0 is any real number. Suppose that $h^n(x_0) \to z$ as $n \to \infty$. Then $h(z) = z$.*

Proof. Suppose that $x_0, h(x_0), h^2(x_0), h^3(x_0), \ldots, h^n(x_0), \cdots \to z$ as $n \to \infty$. Apply h to this sequence. By the continuity of h, we have that $h(x_0), h^2(x_0), \cdots \to h(z)$ as $n \to \infty$ by the continuity of h. However, this latter sequence is just the first sequence without the first term, x_0. So, it must also converge to z, $h(x_0), h^2(x_0), \cdots \to z$. This implies that $h(z) = z$.

□

Let us call this the **Principle of Fixed Point Iteration**. If we iterate a function at a point and the sequence of iterates converges, then it converges to a fixed point of the function. We have a theory for when and why Newton-Raphson Iteration converges, but even if we did not have such a theory, if it converged, the result would result in a fixed point for $g(x)$ and hence a solution for $f(x) = 0$.

2. Further Analysis of the Error

In this section we will use the **Taylor Series** expansion for $g(x)$ to analyze the error in $g^n(x_0)$. What we show is that the error is approximately *squared* each iteration. This gives us what is called **quadratic convergence** for the algorithm. Suppose that we are at a point that is near z where $g(x) = x - \frac{f(x)}{f'(x)}$ as above and $g(z) = z$. Remember that we also have $g'(z) = 0$. Call the point $z + h$. Then the power series for $g(z + h)$ has the following form.

$$g(z+h) = g(z) + g'(z) \cdot h + \frac{g''(z)}{2} \cdot h^2 + \cdots + \frac{g^{(n)}(z)}{n!} \cdot h^n + \cdots$$

As pointed out above, for the function $g(x)$ this simplifies to the following form.

$$g(z+h) = z + 0 \cdot h + \frac{g''(z)}{2} \cdot h^2 + o(h^2)$$

So, if $x_0 = z + h$, then the initial error of this estimate of z is h. However, the error of $g(x_0) = g(z+h)$ is approximately $\frac{g''(z)}{2} \cdot h^2$ which is approximately h^2. If $z+h$ has n digits correct, then $h < 10^{-n}$ and $h^2 < 10^{-2n}$. This implies that $g(z + h)$ has approximately $2n$ digits correct.

3. Example and TI-Nspire CX CAS Program

In this section we give a screenshot of a program for the Newton-Raphson algorithm with an example.

FIGURE 2. Screenshot of the program NewtonRaphson with an example