NUMERICAL ANALYSIS PROBLEMS

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The problems that follow illustrate the methods covered in class. They are typical of the types of problems that will be on the tests.

1. Solving Equations

Problem 1. Suppose that $f : R \to R$ is continuous and suppose that for $a < b \in R$, $f(a) \cdot f(b) < 0$. Show that there is a c with $a < c < b$ such that $f(c) = 0$.

Problem 2. Solve the equation $x^5 - 3x^4 + 2x^3 - x^2 + x = 3$. Solve using the Bisection method. Solve using the Newton-Raphson method. How many solutions are there?

Problem 3. Solve the equation $x = \cos x$ by the Bisection method and by the Newton-Raphson method. How many solutions are there? Solve the equation $\sin(x) = \cos x$ by the Bisection method and by the Newton-Raphson method. How many solutions are there?

Problem 4. Let h be a continuous function $h : R^n \to R^n$. Let $x_0 \in R^n$. Suppose that $h^n(x_0) \to z$ as $n \to \infty$. Show that $h(z) = z$.

Problem 5. Solve the equation $x^4 = 2$ by the Newton-Raphson method. How many real solutions are there? For which starting values x_0 will the method converge?

Problem 6. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and that $f(z) = 0$. Suppose that $f'(z) \neq 0$. Let $g(x) = x - \frac{f(x)}{f'(x)}$. Show that there is an $\varepsilon > 0$ such that for any $x_0 \in [z - \varepsilon, z + \varepsilon]$, $g^n(x_0) \to z$ as $n \to \infty$

Problem 7. Show that the Newton-Raphson method converges quadratically. That is, show that for large n, once there are n digits correct, the next iteration has approximately $2n$ digits correct.

2. Lagrange Polynomials

Problem 8. Determine the polynomial $p(x)$ of degree 5 passing through the points $\left\{(0,0), \left(\frac{1}{2},0\right), (1,0), \left(\frac{3}{2},1\right), (2,0), \left(\frac{5}{2},0\right)\right\}$.

Problem 9. Determine the VanderMonde matrix for the points $\left[0, \frac{1}{9}, \frac{2}{9}, \ldots, 1\right]$.

3. Numerical Integration

Problem 10. Determine the closed Newton-Cotes coefficients for eleven points, $\{a_0, a_1, \ldots, a_{10}\}$. Use these values to estimate the integral

$$\int_{-4}^{4} \frac{1}{1+x^2} dx.$$

Problem 11. Suppose that $\{x_i\}_{i=0}^{n}$ is a set of points in R such that $x_i \neq x_j$ for all $i \neq j$. Let $j_0 \in \{0, 1, \ldots, n\}$. Give a formula for a polynomial $p(x)$ such that $p(x)$ has degree n and such that $p(x_j) = 0$ for $j \neq j_0$ and $p(x_{j_0}) = 1$.

Problem 12. Estimate $\int_0^{\sqrt{\pi}} \sin(x^2) dx$ using Gaussian quadrature.

Problem 13. Show that Gaussian quadrature using $n+1$ points is exact for polynomials of degree $k \leq 2n+1$.

Problem 14. Explain the Romberg method for approximating the integral. If the interval is divided into 2^n subintervals and the Romberg method is applied, what is the error of the method?

Problem 15. Consider the points $\left\{x_0 = \frac{1}{2}, x_1 = \frac{3}{4}, x_2 = \frac{4}{5}\right\}$ in $[0,1]$. What should $\{a_0, a_1, a_2\}$ be so that the estimate $\int_0^1 f(x)dx \approx a_0 \cdot f(x_0) + a_1 \cdot f(x_1) + a_2 \cdot f(x_2)$ is exact for $f(x)$ a polynomial of degree $k \leq 2$?

Problem 16. Consider the points $\left\{x_0 = \frac{\pi}{2}, x_1 = \frac{3\pi}{4}\right\}$ in $[0, \pi]$. What should $\{A_0, A_1\}$ be so that the estimate $\int_0^{\pi} f(x)dx \approx A_0 \cdot f(x_0) + A_1 \cdot f(x_1)$ is exact for $f(x)$ all polynomials of degree $k \leq 1$?

Solution. Let let $f(x)$ be a function on $[0, \pi]$. Then the estimate will be $\int_0^\pi p(x)dx$ where $p(x)$ is the Lagrange polynomial which is $f\left(\frac{\pi}{2}\right)$ at $\frac{\pi}{2}$ and $f\left(\frac{3\pi}{4}\right)$ at $\frac{3\pi}{4}$. Now $p(x) = f\left(\frac{\pi}{2}\right) \cdot p_0(x) + f\left(\frac{3\pi}{4}\right) \cdot p_1(x)$ where $p_0(x) = \frac{\left(x - \frac{3\pi}{4}\right)}{\left(\frac{\pi}{2} - \frac{3\pi}{4}\right)}$ and $p_1(x) = \frac{\left(x - \frac{\pi}{2}\right)}{\left(\frac{3\pi}{4} - \frac{\pi}{2}\right)}$. Now $\int_0^\pi p(x)dx = \int_0^\pi \left(f\left(\frac{\pi}{2}\right) \cdot p_0(x) + f\left(\frac{3\pi}{4}\right) \cdot p_1(x)\right) dx$. This shows that $A_0 = \int_0^\pi p_0(x)dx$ and $A_1 = \int_0^\pi p_1(x)dx$. Thus, $A_0 = \pi$ and $A_1 = 0$.

Problem 17. Give the Legendre polynomials up to degree 10. List the properties that determine these polynomials.

4. Numerical Differentiation

Problem 18. Determine the coefficients to compute the first derivative of $f(x) = \sin(x^2)$ at $a = 2$ using the points $\{a-2h, a-h, a, a+h, a+2h\}$. Give the estimate of the derivative as a function of h. Determine the best value of h for the greatest accuracy of the answer. How many digits accuracy can you expect with this choice of h?

Problem 19. Determine the coefficients to compute the second and third derivative of $f(x) = \sin(x^2)$ at $a = 2$ using the points $\{a-2h, a-h, a, a+h, a+2h\}$. Give the estimate of the second and third derivatives as functions of h. Determine the best value of h for the greatest accuracy of the answer. How many digits accuracy can you expect with this choice of h?

Problem 20. Suppose that $k \leq n$. Show that when estimating the kth derivative of $f(x)$ at a using the points $\{a + m_0 \cdot h, a + m_1 \cdot h, a + m_2 \cdot h, \cdots, 1 + m_n \cdot h\}$, the result is exact $f(x)$ a polynomial of degree $p \leq n$.

5. Differential Equations

Problem 21. Solve the differential equation for $\frac{dx}{dt} = f(t, x) = t \cdot x^2$ with $x(0) = 1$. Solve using Picard iteration for five iterations. Solve using the Taylor method of order 3,4, and 5. Solve using the Euler method, modified Euler, Heun, and Runge-Kutta methods using $h = \frac{1}{20}$ and $n = 20$. Compare the answers and the errors for each of these methods.

Problem 22. How would you go about solving the differential equation $\frac{d^2x}{dt^2} = -x$ with $x(0) = 1$ and $x'(0) = 1$ with each of the methods listed in the previous problem. What changes would need to be made in the programs?

Problem 23. Find a Taylor expansion for the solution $x(t) = a_0 + a_1 t + a_2 t^2 + \cdots$ for the differential equation $\frac{dx}{dt} = t \cdot x$ with the boundary condition $x(0) = 1$. Solve for $\{a_0, a_1, a_2, a_3, a_4, a_5\}$. Do this by hand solving for these coefficients recursively. Solve for the coefficients using the Taylor Method program included in your program collection. Can you determine the general a_n?

6. Simulation and Queueing Theory

Problem 24. Explain the basis for the **Bowling** program. Run some examples with different values for the probability of a strike, spare, and open frame for each frame. Discuss the results.

Problem 25. State the three assumptions for a Poisson process. For a Poisson process with rate α derive the probability of k events in an interval of length t. Show that this is $\Pr[k] = \frac{(\alpha \cdot t)^k}{k!} \cdot e^{-\alpha \cdot t}$.

Problem 26. Give the Kolmogoroff-Chapman equations for a queue with a single server assuming an arrival rate α and service rate σ assuming Poisson arrivals and exponential service times. Assume that $\sigma > \alpha > 0$. The Kendall symbolism for a single server queue with classic assumptions is M/M/1/FIFO.

Problem 27. Assume a queueing system with Poisson arrival rate of α and a single server with an exponential service rate σ. Assume that $\sigma > \alpha > 0$. Determine the steady-state probabilities, $\{\overline{p}_n\}_{n=0}^{\infty}$ for this system.

Problem 28. Assume a queueing system with Poisson arrival rate of α and a single server with an exponential service rate σ. Assume that $\sigma > \alpha > 0$. This is an M/M/1/FIFO queue. Determine the steady-state probabilities for n, $\{\overline{p}_n\}_{n=0}^{\infty}$ for this system. Determine the expected number of customers in the system, $\mathbb{E}[n] = \overline{n} = \sum_{n=0}^{\infty} n\overline{p}_n$.

Problem 29. Use the **Queue** program to simulate a queueing system for M/M/1/FIFO with $\alpha = 9$ and $\sigma = 10$. Simulate a queueing system for M/M/2/FIFO with $\alpha = 9$ and $\sigma = 10$. How do the results compare with the theoretical calculations for $\{\overline{p}_n\}_{n=0}^{\infty}$ in each of these cases?

Problem 30. Suppose that α_1 and α_2 are rates of two independent Poisson processes. Show that the combined process has rate $\alpha_1 + \alpha_2$.

Problem 31. Assume that you have a program that will generate a sequence of independent random numbers from the uniform distribution on $[0,1]$. Your calculator has a program that is purported to have this property. It is the **rand()** function. Determine a program that will generate independent random numbers from the exponential waiting time with parameter α. The probability density function for this waiting time is given by $f(t) = \alpha \cdot e^{-\alpha \cdot t}$ and the cumulative distribution function is given by $F(t) = 1 - e^{-\alpha \cdot t}$.

Problem 32. Suppose that there are an infinite number of servers in the queueing system M/M/∞. Suppose that the arrival rate is α and the service rate for each server is σ. Determine the steady-state probabilities $\{\overline{p}_n\}_{n=0}^{\infty}$ for this system. Explain how this could be used to model the population of erythrocytes in human blood. What would α and σ be in this case? Determine approximate numerical values for α and σ in this case.

Problem 33. In **Gambler's Ruin** two players engage in a game of chance in which A wins a dollar from B with probability p and B wins a dollar from A with probability $q = 1-p$. There are N dollars between A and B and A begins the n dollars. They continue to play the game until A or B has won all of the money. What is the probability that A will end up with all the money assuming that $p > q$. Assume that $p = \frac{20}{38}$ which happens to be the house advantage in roulette. What is the probability that A will win all the money if $n = \$100$ and $N = \$1,000,000,000$?

Problem 34. Suppose that Urn I is chosen with probability $\frac{1}{2}$ and Urn II is also chosen with probability $\frac{1}{2}$. Suppose that Urn I has 5 white balls and 7 black balls and Urn II has 8 white balls and 3 black balls. After one of the urns is chosen, a ball is chosen at random from the urn. What is the probability that the urn was Urn I given that the ball chosen was white?

Problem 35. A test for a disease is positive with probability .95 when administered to a person with the disease. It is positive with probability .03 when administered to a person not having the disease. Suppose that the disease occurs in one in a million persons. Suppose that the test is administered to a person at random and the test is positive. What is the probability that the person has the disease.

Problem 36. Let $f : [a,b] \to [a,b]$ be continuous. Show that $\frac{1}{n} \cdot \sum_{i=1}^{n} f((b-a) \cdot \mathbf{rand()} + a) \cdot (b-a)$ converges to $\int_a^b f(x)dx$ as $n \to \infty$. This limit is the basic underlying principle of **Monte-Carlo simulation**.