Solution of Ordinary Differential Equations

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1 General Theory

Here we give a proof of the existence and uniqueness of a solution of ordinary differential equations satisfying certain conditions. The conditions are fairly minimal and usually satisfied for applications in physics and engineering. There are physical situations where the conditions are not satisfied. In those situations one may not be able to predict the path that the physical system will follow. The differential equation and initial value are given below.

$$\frac{dx}{dt} = f(t,x)$$
$$x(t_0) = x_0$$

Now suppose that $f(t,x)$ is continuous on $D = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b]$ and that $f(t,x)$ satisfies the Lipschitz condition $|f(t,x_2) = f(t,x_1)| \leq K \cdot |x_2 - x_1|$ on D for some $K > 0$. Under these hypotheses there is a positive d and a unique solution $x(t)$ on the interval $[t_0 - d, t_0 + d]$ satisfying $x(t_0) = x_0$.

The first step in proving this result is to transform the differential equation into an integral equation.

$$x(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau))\, d\tau$$

2 Picard Iteration

By thinking of the right hand side of this equation as an operator, the problem now becomes one of finding a fixed point for the integral operator. We apply the *Contraction Mapping Theorem* to argue that the following sequence converges. Let

$$x_2(t) = x_0 + \int_{t_0}^{t} f(\tau, x_1(\tau))\, d\tau$$

and observe the following.

$$|x_2(x) - x_0| \leq M \cdot |t - t_0|$$

This implies that $|x_2(t) - x_0| \leq b$ for $|t - t_0| \leq \min\left\{\frac{b}{M}, a\right\}$. If we define $d = \min\left\{\frac{b}{M}, a\right\}$, then we will prove that there is a unique solution to the differential equation on the interval $[t_0 - d, t_0 + d]$. On this interval define

$$x_{n+1}(t) = x_0 + \int_0^t f(\tau, x_n(\tau))\, d\tau.$$

By induction we will thus obtain an infinite sequence of continuous functions defined on the interval $[t_0 - d, t_0 + d]$. We will now show that this sequence converges by showing that it is Cauchy in the space of continuous functions. First we note the following inequality.

$$\begin{aligned}
|x_{n+1}(t) - x_n(t)| &= \left| \int_{t_0}^t f(\tau, x(\tau))\, d\tau - \int_{t_0}^t f(\tau, x(\tau))\, d\tau \right| \\
&\leq \int_{t_0}^t |f(\tau, x_n(\tau)) - f(\tau, x_{n-1}(\tau))|\ d\tau \\
&\leq K \int_{t_0}^t |x_n(\tau) - x_{n-1}(\tau)|\ d\tau \\
&\leq K\,|t - t_0| \sup\{|x_n(\tau) - x_{n-1}(\tau)| \mid |\tau - t_0| \leq |t - t_0|\}
\end{aligned}$$

This shows that the sequence of iterations $\{x_n(t)\}_{n=0}^{\infty}$ is Cauchy and thus has a limit on the interval $[t - d, t + d]$. Call this limiting function $x(t)$. This gives us the equality.

$$x(t) = x_0 + \int_{t_0}^t f(\tau, x(\tau)) d\tau$$

This implies that $x(t)$ is a solution of the original differential equation on $[t - d, t + d]$ and that it is unique on this interval.

It should be noted that we had to assume that certain conditions were met by $f(t, x)$ in the differential equation. The conditions guaranteed that there would be a unique solution to the differential equation using the fact that Picard Iteration would always converge whatever the initial function $x_0(t)$ for the iteration and that the limiting function would be a solution. If the conditions on the function $f(t, x)$ are not met, then a solution may not exist or there may be many solutions.

3 An Example

Solve the following differential equation $\frac{dx}{dt} = tx$ with $x(0) = 1$ using Picard Iteration.

$$
\begin{aligned}
x_0(t) &\equiv 1 \\
x_2(t) &= 1 + \int_0^t 1 \cdot \tau \, d\tau = 1 + \frac{t^2}{2} \\
x_3(t) &= 1 = +\frac{t^2}{2} + \frac{t^4}{8} \\
x_4(t) &= 1 + \frac{t^2}{2} + \frac{t^4}{8} + \frac{t^6}{48} \\
&\vdots
\end{aligned}
$$

In this case the differential equation is separable. We can solve it easily to get

$$x(t) = \exp\left(\frac{t^2}{2}\right).$$

Picard iteration is giving us the power series of this solution. Each iteration gives us an additional term. This is not always the case as you can see by experimenting with the program in the next section.

4 Implementation on the TI-89

Here is a program that implements Picard Iteration on the TI-89. The variable f is a function with variables t and x. The variables a and b are the initial values t_0 and x_0 such that $x(t_0) = x_0$. The variable n is the number of iterations to be done. To be certain that the steps can be integrated, f should be a polynomial in t and x, but the method will work as long as the functions can be integrated at each step. The program stores the n^{th} iteration in p. To check the program

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picard(t*x,0,1,4)
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into the commandline in the home screen. Then enter p. Stored in p is $1+\frac{t^2}{2}+\frac{t^4}{8}+\frac{t^6}{48}+\frac{t^8}{384}$.

```
:picard(f,a,b,n)
:Prgm
:b → p
:For i,1,n
:p|t=z → p
:b + ∫(f|x=p and t=z,z,a,t) → p
:EndFor
:EndPrgm
```