

POWER SERIES

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1. INTRODUCTION

In this summary we will cover the basic features of power series. Power series is an important way to analyze functions in various settings. When a function agrees with its power series it is said to be **analytic**. Sometimes one specifies the interval over which the function is analytic. The functions with power series representation take us beyond the polynomials. The analytic functions include polynomials, rational functions, the logarithm function, the exponential function, the trigonometric functions, among many others. Virtually every function with major applications is in this category.

2. THE GEOMETRIC SERIES

Of first importance is the **geometric series**. We will prove the convergence of this series in this section. The proof is simple, but it is fundamental in what will follow.

Theorem 1. *Suppose that $|x| < 1$, then*

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Furthermore, if $0 < \varepsilon < \frac{1}{2}$, then $\sum_{n=0}^{\infty} x^n$ converges uniformly to its limit on the interval $[-1 + \varepsilon, 1 - \varepsilon]$.

If $|x| \geq 1$, then the series diverges.

Proof. Assume that $|x| < 1$. Let N be a positive integer and let $S_N = \sum_{n=0}^N x^n$. Then

$$x \cdot S_N = \sum_{n=0}^N x^{n+1} = \sum_{n=1}^{N+1} x^n$$

and

$$(1-x) \cdot S_N = \sum_{n=0}^N x^n - \sum_{n=1}^{N+1} x^n = 1 - x^{N+1}$$

which gives us

$$S_N = \frac{1 - x^{N+1}}{1 - x}.$$

Since $|x| < 1$, $\lim_{n \rightarrow \infty} x^n = 0$ and thus $\lim_{n \rightarrow \infty} S_N = \frac{1}{1-x}$. This is precisely what we mean by

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

For the second part of the proof, let $\delta > 0$ and suppose that $0 < \varepsilon < \frac{1}{2}$. Let $x \in [-1 + \varepsilon, 1 - \varepsilon]$. Let N be such that for all $K \geq N$, $\sum_{n=K}^{\infty} (1 - \varepsilon)^n = \frac{(1 - \varepsilon)^K}{\varepsilon} < \delta$. Now for any $0 \leq |x| < 1 - \varepsilon$ and $K \geq N$,

$$\left| \sum_{n=K}^{\infty} x^n = \frac{x^K}{1-x} \right| \leq \sum_{n=K}^{\infty} |x|^n = \frac{|x|^K}{1-|x|} \leq \sum_{n=K}^{\infty} (1 - \varepsilon)^n = \frac{(1 - \varepsilon)^K}{\varepsilon} < \delta.$$

Thus, $\sum_{n=0}^{\infty} x^n$ converges uniformly to $\frac{1}{1-x}$ on $[-1 + \varepsilon, 1 - \varepsilon]$.

It is easy to show that if $|x| \geq 1$, then $\sum_{n=0}^{\infty} x^n$ diverges. We will leave that to the reader. □

3. GENERAL POWER SERIES AND THE RADIUS OF CONVERGENCE

Suppose that $\sum_{n=0}^{\infty} a_n x^n$ is any power series centered at 0. Let

$$\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \alpha$$

where $0 \leq \alpha \leq \infty$. Let $R = \frac{1}{\alpha}$, $0 \leq R \leq \infty$. Then we have the following theorem.

Theorem 2. *The series $\sum_{n=0}^{\infty} a_n x^n$ converges for all $|x| < R$. Furthermore, it converges uniformly on any interval $[-M, M]$ with $0 \leq M < R$. If $|x| > R$, then the series diverges. For $x = R$ or $x = -R$, the series may converge or it may not converge.*

Proof. The case $R = 0$ is trivial. The proof for R finite can be modified for the case that $R = \infty$. So, we assume that R is finite. Let $|x| < R$. Suppose that N is such that for all $n \geq N$, $\sqrt[n]{|a_n|} < \frac{1}{|x|}$. Note that $\frac{1}{|x|} > \alpha = \frac{1}{R}$. So, for all $n \geq N$, $\sqrt[n]{|a_n|} |x|^n < \frac{|x|}{R}$. Of course, this implies that

$$\left(\sqrt[n]{|a_n|} |x|^n \right)^n < \left(\frac{|x|}{R} \right)^n$$

and thus that

$$\sum_{n=N}^{\infty} \left(\sqrt[n]{|a_n|} |x|^n \right)^n = \sum_{n=N}^{\infty} |a_n| |x|^n < \sum_{n=N}^{\infty} \left(\frac{|x|}{R} \right)^n$$

converges. This implies that $\sum_{n=N}^{\infty} a_n x^n$ converges for $|x| < R$. This implies that $\sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < R$ since this last sum adds only a finite number of terms to the series.

The fact that this series converges uniformly on any interval $[-M, M]$ with $0 \leq M < R$ follows from the proof for that case in the proof of Theorem 1.

The fact that the series diverges for $|x| > R$ is easy to prove and we leave it as an exercise for the reader. There are examples where the series may converge or diverge at $x = R$ or $x = -R$. \square

We say that R as defined above is the **radius of convergence** for the power series $\sum_{n=0}^{\infty} a_n x^n$. Note that the radius of convergence R can be characterized as the largest R such that the series converges for all $x \in (-R, R)$ and diverges for $|x| > R$. The formula $\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \alpha$ with $R = \frac{1}{\alpha}$ gives us a way to calculate the value of R , but if we could determine a value M by any means such the series converges for all $x \in (-M, M)$ and diverges for $|x| > M$, then M would be the radius of convergence for the series. In §9 we will give another way which is sometimes useful in determining the radius of convergence.

4. ANTIDERIVATIVES AND INTEGRATION OF POWER SERIES

Theorem 3. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and suppose that the radius of convergence for this series is $R > 0$. Let $g(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$. Then the radius of convergence for $g(x)$ is also R and for all $|x| < R$, $g'(x) = f(x)$.

Proof. Consider the power series $\sum_{n=0}^{\infty} a_n x^n$ with radius of convergence R . Note that if $R = 0$, then the series only converges for $x = 0$. So, for a meaningful discussion, we have assumed that $0 < R \leq \infty$. Now $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is well-defined on the interval $(-R, R)$. Let $g(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$. It is easy to check that $g(x)$ is represented by a power series that also has radius of convergence R . Let $|x| < R$. Then note that

$$\begin{aligned}
 (1) \quad \int_0^x f(t) dt &= \int_0^x \left(\sum_{n=0}^{\infty} a_n t^n \right) dt = \int_0^x \lim_{N \rightarrow \infty} \left(\sum_{n=0}^N a_n t^n \right) dt \\
 (2) \quad &= \lim_{N \rightarrow \infty} \int_0^x \left(\sum_{n=0}^N a_n t^n \right) dt \\
 (3) \quad &= \lim_{N \rightarrow \infty} \sum_{n=0}^{N+1} \frac{a_{n+1}}{n+1} x^n \\
 (4) \quad &= \sum_{n=0}^{\infty} \frac{a_{n+1}}{n+1} x^n \\
 (5) \quad &= g(x)
 \end{aligned}$$

Going from the first to the second step is because the convergence of $\sum_{n=0}^{\infty} a_n t^n$ is uniform on the interval over which the integration takes place. So, this proves that $g(x)$ is an antiderivative of $f(x)$ and that $g'(x) = f(x)$. \square

We can apply this theorem to several examples. We know from elementary Calculus that $\frac{d}{dx} \ln(1+x) = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n$. By integrating term by term as above we get

$$\ln(1+x) + C = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

for all $|x| < 1$ for some constant C . By evaluating both sides of the equation at $x = 0$ we obtain that $C = 0$. Thus,

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}.$$

Similarly, $\frac{d}{dx} \arctan(x) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n$. Again, by integrating term by term we get

$$\arctan(x) + C = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n+1}}{2n+1}$$

for some constant C for all $|x| < 1$. By evaluating both sides of the equation at $x = 0$ we get that $C = 0$. Thus,

$$\arctan(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n+1}}{2n+1}.$$

5. DERIVATIVES OF POWER SERIES

Now just because the power series $\sum_{n=0}^{\infty} a_n x^n$ converges on $(-R, R)$ and even uniformly on $[-M, M] \subset (-R, R)$ does not mean that the series which is the term-by-term derivative of that series would necessarily be the derivative of the series. It turns out that this is the case, but it requires a proof. The proof that we give is indirect.

Theorem 4. *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and suppose that $f(x)$ has radius of convergence R . Let $g(x) = \sum_{n=1}^{\infty} (n+1)a_{n+1}x^n$. Then $g(x)$ also has radius of convergence R and $f'(x) = g(x)$ for all $x \in (-R, R)$.*

Proof. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and let $g(x) = \sum_{n=1}^{\infty} (n+1)a_{n+1}x^n$. Note that $g(x)$ is the term by term derivative of $f(x)$.

Suppose that R is the radius of convergence for the series that defines $f(x)$. Then

$$\frac{1}{R} = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \alpha.$$

However, it is easy to show that

$$\frac{1}{R} = \limsup_{n \rightarrow \infty} \sqrt[n]{(n+1)|a_{n+1}|} = \alpha$$

as well. So, the series representing $f(x)$ and that representing $g(x)$ both have radius of convergence R . Since $f(x)$ is the term by term integral of $g(x)$, it must be that it is the antiderivative of $g(x)$. Thus, $f'(x) = g(x)$ as was to be proved. \square

We have examples of this as well. Since the Geometric Series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ has radius of convergence $R = 1$,

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n$$

also has radius of convergence $R = 1$.

6. POWER SERIES CENTERED AT AN ARBITRARY POINT

We have been discussing power series centered at 0. We could have defined a power series centered at a for any real a . This would take the form

$$\sum_{n=0}^{\infty} a_n(x-a)^n.$$

We calculate the radius of convergence the same way as in §3 using the coefficients a_n of the series. If the radius of convergence is R , then the series centered at a will converge in the interval $(a-R, a+R)$.

7. THE COEFFICIENTS

Theorem 5. *Let $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ with radius of convergence $R > 0$. Then the coefficients of the series must satisfy $a_n = \frac{f^{(n)}(a)}{n!}$.*

Proof. The derivative of $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ is another power series with the same radius of convergence R . Thus $f(x)$ is infinitely differentiable at every point $x \in (a-R, a+R)$. In particular, $f^{(n)}(a)$ exists for every n . However,

$$f'(x) = \sum_{n=0}^{\infty} (n+1)a_{n+1}(x-a)^n.$$

When we evaluate $f'(x)$ at $x = a$, we get $f'(a) = a_1$. By induction we get that $f^{(n)}(a) = n!a_n$ and thus that

$$a_n = \frac{f^{(n)}(a)}{n!}.$$

□

8. GENERAL ANALYTIC FUNCTIONS – THE TAYLOR REMAINDER THEOREM

If we have a given function, say $\sin(x)$, that we imagine is analytic, we can determine the power series that should represent it by determining the coefficients from the previous section. In the case of $\sin(x)$ and $a = 0$ we get that

$$a_n = \begin{cases} 0 & n = 0 \pmod{4} \\ \frac{1}{n!} & n = 1 \pmod{4} \\ 0 & n = 2 \pmod{4} \\ \frac{-1}{n!} & n = 3 \pmod{4} \end{cases}$$

We thus get a power series.

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

We can easily check that the radius of convergence of the series is $R = \infty$. The easiest way is to use the Ratio Test in §8. However, does this power series represent the function $\sin(x)$? That is, can we say

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

And, does the power series represent $\sin(x)$ for all x or perhaps just some x ?

For a few functions we can determine the answer to this by integrating or differentiating functions having known power series as we did for $\ln(1+x)$ and $\arctan(x)$ in §4 and $\frac{1}{(1-x)^2}$ §5. However, for most functions we need to use the following **Taylor Remainder Theorem** to determine when the series converges to the given function. We state the Taylor Remainder Theorem below. Its proof is given at another link.

Theorem 6. *Suppose that $f(x)$ is $(N+1)$ times differentiable on the interval $[a, b]$ with $a < x_0 < b$. Let $a < x_0 < b$. Then there is a point ξ between x_0 and x such that the following holds.*

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x-x_0) + \frac{f''}{2}(x-x_0)^2 + \cdots + \frac{f^{(N)}(x_0)}{N!}(x-x_0)^N + \frac{f^{(N+1)}(\xi)}{(N+1)!}(x-x_0)^{N+1} \\ &= \sum_{n=0}^N \frac{f^{(n)}}{n!}(x-x_0)^n + \frac{f^{(N+1)}(\xi)}{(N+1)!}(x-x_0)^{(N+1)} \end{aligned}$$

Let us show how to apply this technique to show that

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}.$$

We are letting $x_0 = 0$ in Theorem 6 in our application. For a truncated series, the theorem will take the following form.

$$\sin(x) = \sum_{k=0}^K \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \frac{\frac{d^{2K+2}}{dx^{2K+2}} \sin(x) \Big|_{x=\xi}}{(2K+2)!} x^{2K+2}.$$

However, $\left. \frac{d^{2K+2}}{dx^{2K+2}} \sin(x) \right|_{x=\xi}$ is just $\pm \sin(\xi)$ and the absolute value of this is at most 1. So, the remainder term is given by

$$\left| \frac{\left. \frac{d^{2K+2}}{dx^{2K+2}} \sin(x) \right|_{x=\xi} x^{2K+2}}{(2K+2)!} \right| \leq \frac{|x|^{2K+2}}{(2K+2)!}.$$

It can be easily shown that $\frac{|x|^{2K+2}}{(2K+2)!} \rightarrow 0$ as $K \rightarrow \infty$.

The technique can be used to show that the power series for $\sin(x)$ and $\cos(x)$ converge when centered at any point a . The trick is to show that the remainder term in the Taylor Remainder Theorem goes to zero as $N \rightarrow \infty$ whatever the ξ may be.

9. THE RATIO TEST

The general method for computing the radius of convergence of a power series was given in §3. In this section we give another method for determining convergence of a power series. This is known as the **Ratio Test**. Suppose that we have a power series of the form $\sum_{n=0}^{\infty} a_n x^n$.

Theorem 7. *Suppose that*

$$\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = \beta.$$

Then $\sum_{n=0}^{\infty} a_n x^n$ converges for all $|x| < \rho = \frac{1}{\beta}$.

Proof. Assume the hypotheses of Theorem 7. Let $|x| < \rho$. Let $\delta > 0$ be such that $|x| < \frac{1}{\beta+2\delta} < \frac{1}{\beta} = \rho$. Let N be such that for all $n \geq N$, $\frac{|a_{n+1}|}{|a_n|} < \beta + \delta$. Then

$$\frac{|a_{n+1}x^{n+1}|}{|a_n x^n|} = \frac{|a_{n+1}x|}{|a_n|} < \frac{\beta + \delta}{\beta + 2\delta} < 1.$$

This implies that

$$\left| \sum_{n=N}^{\infty} a_n x^n \right| \leq \sum_{n=N}^{\infty} |a_n x^n| < |a_N| \sum_{n=0}^{\infty} \left(\frac{\beta + \delta}{\beta + 2\delta} \right)^n < \frac{1}{1 - \frac{\beta + \delta}{\beta + 2\delta}} = \frac{\beta + 2\delta}{\delta}.$$

This implies that our series $\sum_{n=0}^{\infty} a_n x^n$ converges.

Note that under the assumptions of Theorem 7, if $x > \frac{1}{\beta}$, then the series does not converge since the individual terms do not go to zero. This implies that $\rho = \frac{1}{\beta}$ is the radius of convergence for the series.

This is a useful way to determine the radius of convergence for many series. However, it only works if the limit $\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = \beta$ exists. □