THE RIEMANN INTEGRAL

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The purpose of this document is to give a brief summary of the Riemann integral. We start with the definition.

1. Riemann integral and its existence

Definition. Suppose that $f : [a, b] \to \mathbb{R}$ is a bounded function. The *Riemann integral* of $f(x)$ is denoted by $\int_a^b f(x)dx$. Let $\mathscr{P} = \{x_0 = a < x_1 < x_2 < \cdots < x_n = b\}$ be a partition of $[a, b]$. For each i, let $M_i = \sup_{x\in[x_{i-1},x_i]} f(x)$ and $m_i = \inf_{x\in[x_{i-1},x_i]} f(x)$. Let

$$\overline{S}_{\mathscr{P}} = \sum_{i=1}^{n} M_i \Delta x_i$$

and

$$\underline{S}_{\mathscr{P}} = \sum_{i=1}^{n} m_i \Delta x_i.$$

Let

$$\underline{\int_a^b} f(x)dx = \sup_{\mathscr{P}} \underline{S}_{\mathscr{P}} = \sum_{i=1}^{n} m_i \Delta x_i$$

and

$$\overline{\int_a^b} f(x)dx = \inf_{\mathscr{P}} \overline{S}_{\mathscr{P}} = \sum_{i=1}^{n} M_i \Delta x_i.$$

Note that $\underline{\int_a^b} f(x)dx \leq \overline{\int_a^b} f(x)dx$. We say that $f(x)$ is *Riemann integrable* over $[a, b]$ when $\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx$ and denote the common value by $\int_a^b f(x)dx$.

Theorem 1. *Suppose that $f(x)$ is non-decreasing on $[a, b]$. Then $f(x)$ is Riemann integrable over $[a, b]$.*

Proof. Let n be a positive integer and let

$$\mathscr{P}_n = \left\{x_0 = a, x_1 = a + \frac{(b-a)}{n}, x_2 = a + \frac{2\cdot(b-a)}{n}, \ldots, x_n = b\right\}.$$

Note that $M_i = f(x_i)$ and $m_i = f(x_{i-1})$. So,

$$\underline{S}_{\mathscr{P}} = \sum_{i=1}^{n} f(x_{i-1})\Delta x_i$$

and

$$\overline{S}_{\mathscr{P}} = \sum_{i=1}^{n} f(x_i)\Delta x_i.$$

One can easily see that as $n \to \infty$,

$$\overline{S}_{\mathscr{P}} - \underline{S}_{\mathscr{P}} = \frac{(f(b) - f(a)) \cdot (b-a)}{n} \to 0.$$

$\square$

Theorem 2. *Suppose that $f(x)$ is continuous on $[a,b]$. Then $f(x)$ is Riemann integrable over $[a,b]$.*

Proof. Let $\varepsilon > 0$. Since $f(x)$ is continuous on $[a,b]$ and $[a,b]$ is compact, $f(x)$ must be uniformly continuous. Let $\delta > 0$ be such that whenever $|x-y| < \delta$, $|f(x) - f(y)| < \varepsilon$. Let n be a positive integer such that $\frac{b-a}{n} < \delta$. Then let

$$\mathscr{P}_n = \left\{ x_0 = a, x_1 = a + \frac{(b-a)}{n}, x_2 = a + \frac{2 \cdot (b-a)}{n}, \ldots, x_n = b \right\}.$$

Then

$$\overline{S}_{\mathscr{P}} = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} M_i \cdot \frac{(b-a)}{n}$$

and

$$\underline{S}_{\mathscr{P}} = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} m_i \cdot \frac{(b-a)}{n}.$$

Thus,

$$\overline{S}_{\mathscr{P}} - \underline{S}_{\mathscr{P}} = \sum_{i=1}^{n} (M_i - m_i) \cdot \frac{(b-a)}{n} < n \cdot \varepsilon \cdot \frac{(b-a)}{n} = \varepsilon \cdot (b-a).$$

Thus, we can conclude that

$$\overline{\int_a^b} f(x)dx = \inf_{\mathscr{P}} \overline{S}_{\mathscr{P}} = \sup_{\mathscr{P}} \underline{S}_{\mathscr{P}} = \underline{\int_a^b} f(x)dx.$$

$\square$

Through the proof it is easy to see that we could use the upper sum or the lower sum in either of the above theorems and the analysis in the proofs to estimate how close these sums are to the true integral. This would be a rather crude numerical estimate. We can estimate the value of the integral much more accurately using Romberg integration. We describe this method in a separate posting.

2. The Fundamental Theorem of Calculus

Theorem 3 (The Fundamental Theorem of Calculus)**.** *Suppose that $f(x)$ is continuous on $[a, b]$. Suppose that $F(x)$ is differentiable on $[a, b]$ with $F'(x) \equiv f(x)$ on $[a, b]$. Then $\int_a^b f(x)dx = F(b) - F(a)$.*

Proof. Define $F(x) = \int_a^x f(t)dt$. Then $F(x)$ is defined for all $x \in [a, b]$. We now show that $F'(x) \equiv f(x)$.

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \frac{\int_x^{x+h} f(t)dt}{h}$$

We want to show that the limit above is equal to $f(x)$ for each x. Let $\varepsilon > 0$ be given. By the continuity of $f(x)$ at x, let $\delta > 0$ be such that if $|y - x| < \delta$, then $|f(y) - f(x)| < \varepsilon$. Then if $|h| < \delta$, then

$$\frac{h \cdot (f(x) - \varepsilon)}{h} = f(x) - \varepsilon < \frac{\int_x^{x+h} f(t)dt}{h} < \frac{h \cdot (f(x) + \varepsilon)}{h} = f(x) + \varepsilon.$$

Thus,

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \frac{\int_x^{x+h} f(t)dt}{h} = f(x)$$

as required.

To complete the proof of the Fundamental Theorem of Calculus, let $G(x)$ be any differentiable function on $[a, b]$ such that $G'(x) \equiv f(x)$. Then let $h(x) = G(x) - F(x)$. Then $h'(x) \equiv 0$ by the Mean Value Theorem. Thus, there is a constant C such that $h(x) \equiv C$. Thus $G(x) \equiv F(x) + C$. Thus,

$$\int_a^b f(x)dx = F(b) - F(a) = G(b) - G(a).$$

□

Note that if the Riemann integral exists for $f(x)$ over $[a, b]$, then $F(x) = \int_a^x f(t)dt$ exists for all $a \leq x \leq b$. No assumption about the continuity of $f(x)$ is necessary for this existence. However, $F'(x)$ may not exist at a point x. Nevertheless, if f is continuous at x, then F' exists at x and $F'(x) = f(x)$ at this x.