Romberg Integration

Trapezoidal Rule: The formula for the Trapezoidal Rule is as follows:

$$T(n) = \frac{b-a}{2n}(f(x_0) + 2f(x_1) + 2f(x_2) \cdots 2f(x_{n-1}) + f(x_n))$$

The Trapezoidal Rule approximates the integral in this fashion.

$$T(n) = \int_a^b f(x)dx + a_1\left(\tfrac{1}{n^2}\right)^1 + a_2\left(\tfrac{1}{n^2}\right)^2 + a_3\left(\tfrac{1}{n^2}\right)^3 + \cdots$$

The terms following the integral are the *error terms*. Romberg integration eliminates successive terms in the error in the formula.

Romberg Integration: The Romberg method removes successive error terms. First we compute the following values for $k = 1,2,\ldots,m$, $A_{k,1} = T\left(2^{k-1}\right)$. Then we compute:

$$A_{k,j} = \frac{4^{j-1}A_{k+1,j-1} - A_{k,j-1}}{4^{j-1}-1} \qquad \text{for } j = 2,\ldots,m \text{ and } k = 1,2,\ldots,m-j+1$$

$$\begin{array}{llllll}
A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} & \cdots & A_{1,m} \\
A_{2,1} & A_{2,2} & A_{2,3} & \vdots & \iddots & \\
A_{3,1} & A_{3,2} & \vdots & A_{m-3,4} & & \\
A_{4,1} & \vdots & A_{m-2,3} & & & \\
\vdots & A_{m-1,2} & & & & \\
A_{m,1} & & & & &
\end{array}$$

The error in estimating the integral by $A_{k,j} = \dfrac{4^{j-1}A_{k+1,j-1} - A_{k,j-1}}{4^{j-1}-1}$ is $c_j\left(\frac{1}{n^2}\right)^j + c_{j+1}\left(\frac{1}{n^2}\right)^{j+1} + \cdots$. Below is the Romberg method applied to the integral $\int_0^1 e^{-x^2}dx$. The first column is the Trapezoidal Rule with $n = 1,2,4,8,16,$ and 32 intervals. The other columns are computed as described above. The number at the bottom of each column is the best estimate of the integral in that column. The number furthest to the right in the array is the best estimate overall. It has error $O(h^{12})$ in this example.

.6839397206
.7313702518 .7471804289
.7429840978 .7468553798 .7468337098
.7458656148 .7468261205 .7468241699 .7468240185
.7465845968 .7468242574 .7468241332 .7468241326 .7468241331
.7467642547 .7468241406 .7468241328 .7468241328 .7468241328 .7468241328