Markov Graphs and Sharkovsky's Theorem

Let $f : I \to I$ be a continuous mapping with *I* an interval in the real line. Consider the following ordering of the positive integers.

$$3 \lhd 5 \lhd 7 \lhd \cdots \lhd 2 \cdot 3 \lhd 2 \cdot 5 \lhd 2 \cdot 7 \lhd \cdots \lhd 2^n \cdot 3 \lhd 2^n \cdot 5 \lhd 2^n \cdot 7 \lhd \cdots$$
$$\cdots \lhd 2^{n+1} \cdot 3 \lhd 2^{n+1} \cdot 5 \lhd \cdots \lhd 2^n \lhd 2^{n-1} \lhd \cdots \lhd 8 \lhd 4 \lhd 2 \lhd 1$$

This ordering is known as the *Sharkovsky ordering*. In this ordering if the map *f* has a periodic orbit of period *k* and $k \lhd m$, then *f* also has a periodic orbit of period *m*. This is known as *Sharkovsky's Theorem*.

One basis for proving this theorem is by means of *Markov graphs*. Suppose that $f : I \to I$ is as above and that $\{I_j\}_{j=1}^{p}$ is a collection of subintervals of *I* which are pairwise disjoint except possibly for their endpoints. The Markov graph for this collection of intervals is a directed graph with the vertices being the collection of intervals $\{I_j\}_{j=1}^{p}$ and with an arrow $I_i \to I_j$ precisely when $f(I_i) \supset I_j$.

Theorem. Suppose that $\{I_1 \to I_2 \to \cdots \to I_k\}$ is a cycle in this Markov graph, that is, $I_1 = I_k$. Then there is a point $x \in I_1$ such that $f^k(x) = x$ and $f^i(x) \in I_i$.

Obviously, such a point is periodic with period *n* or less and the period must be an integral factor of *n*. If the cycle is not a multiple of a smaller cycle and if the endpoints of the intervals do not have a period which is a factor of *n*, then *x* must have period *n*. The following two lemmas are essential in proving the Markov graph theorem.

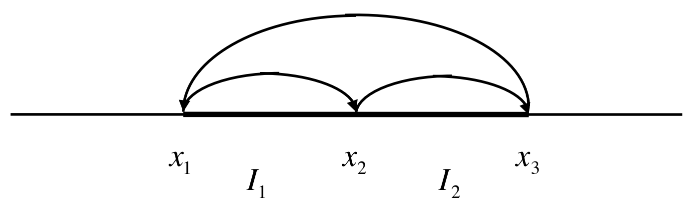
Lemma 1. Suppose that $f : I \to R$ is continuous and that $f(I) \supset I$. Then there is an $x \in I$ such that $f(x) = x$.

Lemma 2. Suppose that $f : I \to R$ is continuous and that $f(I) \supset J$ with *J* an interval. Then there is an interval $K \subset I$ such that $f(K) = J$ and such that no proper subinterval of *K* maps onto *J*.

We will now use the Markov graph theorem to prove that period three implies all other periods. This is a special case of Sharkovsky's Theorem and illustrates the methods of proof for the most general case.

Theorem. Suppose that $f : I \to I$ is continuous and that f has a periodic point of period three. Then for every n there is a periodic point with period n.
Proof. Let $x_1 < x_2 < x_3$ be the period three orbit that f is assumed to have. Assume also that $f(x_1) = x_2$, $f(x_2) = x_3$, and $f(x_3) = x_1$. Let $I_1 = [x_1, x_2]$ and $I_2 = [x_2, x_3]$. Then the Markov graph for these two intervals and a diagram is given below.

$$I_1 \rightleftarrows I_2 \circlearrowleft$$



Now let n be any positive integer not one or three. Consider the cycle $\{I_1 \to I_2 \to \cdots \to I_2 \to I_1\}$ of length n where only the first and last intervals are I_1. Let $x \in I_1$ be the point given by the Markov graph theorem for this cycle. Then $f^n(x) = x$ and $f^i(x) \in I_2$ for all $1 \le i < n$. Now if $f^i(x) = x$ for some $i < n$, then $x \in I_1 \cap I_2$ and thus $x = x_2$ has period three and cannot possibly follow the itinerary of the cycle we assumed. Thus, it cannot happen that $f^i(x) = x$ for some $i < n$ and thus x has period n.

We assumed that $f(x_1) = x_2$, $f(x_2) = x_3$, and $f(x_3) = x_1$. The alternative is that $f(x_1) = x_3$, $f(x_3) = x_2$, and $f(x_2) = x_1$. A similar argument can be made for this case as well to show that period three implies period n for all n. $\square$

Example. Let $f : I \to I$ be the function defined by the following formula.

$$f(x) = \begin{cases} 2x & 0 \le x \le \frac{1}{2} \\ 2 - 2x & \frac{1}{2} \le x \le 1 \end{cases}$$

It is a helpful exercise to use Markov graphs to determine the number of periodic orbits of various periods n. To analyze this problem let $I_1 = \left[0, \frac{1}{2}\right]$ and let $I_2 = \left[\frac{1}{2}, 1\right]$. The Markov graph for these subintervals is the following.

$$\circlearrowright I_1 \rightleftarrows I_2 \circlearrowleft$$

Determine the cycles in this Markov graph that give period n orbits. Also, show that this cycle determines the period n orbit uniquely. Use these results to count the period n orbits. Lastly, come up with an numerical algorithm to find the period n orbit given the cycle in the Markov graph associated with it.

One last comment should be made concerning the existence of the periodic orbits implied by Sharkovsky's Theorem. In the bifurcation diagram produced for the quadratic family of functions $f_\mu = \mu x(1-x)$ for certain values of μ there are attracting periodic orbits. In particular, there is an attracting periodic orbit of period three. Of course, by Sharkovsky's Theorem for these same values of μ there are periodic orbits of all periods. Where are these other periodic orbits? The bifurcation diagram only shows the one period three orbit. It turns out, in this case, that there is only one attracting periodic orbit which in this range is the period three orbit. The basin of attraction of this orbit is dense in the interval and the complement of the basin of attraction contains a Cantor which contains all of the other periodic points except one of the fixed points. Determining the precise location of these other periodic orbits and determining the cardinality of the number of these orbits for each period n is a matter of another discussion, but it has been done.