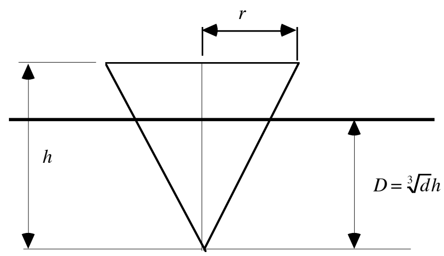
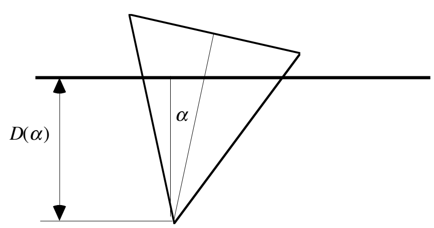
Stability of a Floating Cone with Vertex Down



Assume that the cone is right and has height h and that the base is a circle of radius r. Let the angle of the vertex be denoted by β so that $\tan\left(\frac{\beta}{2}\right) = \frac{r}{h}$. Assume that the density of the cone is d with $0 < d < 1$ with the density of water being 1. Then for the cone to be at equilibrium in the above position, the depth of the vertex will be at $D = \sqrt[3]{d}h$. Now consider the cone slightly tipped at an angle α from vertical.



Assume that the coordinates are at the vertex of the cone with the y-axis vertical and the x-axis horizontal. The center of gravity of the cone so tipped will have the following x-coordinate.

$$x_2(\alpha) = \frac{3}{4} h \sin(\alpha)$$

The center of gravity of the displaced water will have the following x-coordinate.

$$x_1(\alpha) = \frac{3}{8} D(\alpha)\left[\tan\left(\tfrac{\beta}{2} + \alpha\right) - \tan\left(\tfrac{\beta}{2} - \alpha\right)\right]$$

This last result was obtained by noting that the displaced water will form a cone with base an ellipse. The length of the major axis of the ellipse will be $L = D(\alpha)\left[\tan\left(\tfrac{\beta}{2} - \alpha\right) + \tan\left(\tfrac{\beta}{2} + \alpha\right)\right]$. The centroid of this ellipse will be at the midpoint of the major axis and the centroid of the displaced water will be on the line joining this point to the vertex of the cone and $\frac{3}{4}$ of the distance from the vertex to the centroid of the elliptical face.

To determine stability we compute the following limit.

$$\lim_{\alpha \to 0} \frac{x_1(\alpha)}{x_2(\alpha)}$$

The vertical position is stable if this limit is greater than one and unstable if this limit is less than one. This limit is computed below.

$$\lim_{\alpha \to 0} \frac{\frac{3}{8} D(\alpha)\left[\tan\left(\frac{\beta}{2}+\alpha\right)-\tan\left(\frac{\beta}{2}-\alpha\right)\right]}{\frac{3}{4} h \sin(\alpha)}$$

This limit is easily seen to be

$$\frac{D(0)}{2h} \lim_{\alpha \to 0} \frac{\left(\tan\left(\frac{\beta}{2}+\alpha\right)-\tan\left(\frac{\beta}{2}-\alpha\right)\right)}{\sin(\alpha)}$$

Since $D(0)=\sqrt[3]{d}h$ and the right hand limit is $2\sec^2\left(\frac{\beta}{2}\right)$, we have stability precisely when

$$\cos^6\left(\tfrac{\beta}{2}\right) > d\,.$$

A similar argument establishes the stability for a long right cylinder with with isosceles triangular base whose density is *d*. The angle β of the vertex of the isosceles triangular cross-section must be

$$\cos^4\left(\tfrac{\beta}{2}\right) > d\,.$$