THE TAYLOR REMAINDER THEOREM

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In this post we give a proof of the Taylor Remainder Theorem. It is a very simple proof and only assumes Rolle's Theorem.

Rolle's Theorem. Let $f(x)$ be differentiable on $[a, b]$ and suppose that $f(a) = f(b)$. Then there is a point $a < \xi < b$ such that $f'(\xi) = 0$.

Taylor Remainder Theorem. Suppose that $f(x)$ is $(N + 1)$ times differentiable on the interval $[a, b]$ with $a < x_0 < b$. Let $a < x_0 < b$. Then there is a point ξ between x_0 and x such that the following holds.

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''}{2}(x - x_0)^2 + \cdots + \frac{f^{(N)}(x_0)}{N!}(x - x_0)^N + \frac{f^{(N+1)}(\xi)}{(N+1)!}(x - x_0)^{N+1} \\ &= \sum_{n=0}^{N} \frac{f^{(n)}}{n!}(x - x_0)^n + \frac{f^{(N+1)}(\xi)}{(N+1)!}(x - x_0)^{N+1} \end{aligned}$$

Proof. Let f, a, b, x and x_0 be as in the statement of the theorem. Let R be defined by the following equation.

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''}{2}(x - x_0)^2 + \cdots + \frac{f^{(N)}(x_0)}{N!}(x - x_0)^N + \frac{R}{(N+1)!}(x - x_0)^{N+1}$$

Define $F(\xi)$ by the following formula.

$$F(\xi) = \sum_{n=0}^{N} \frac{f^{(n)}(\xi)}{n!}(x - \xi)^n + \frac{R}{(N+1)!}(x - \xi)^{N+1}$$

Then we have the following.

$$\begin{aligned} F'(\xi) &= f'(\xi) + \sum_{n=0}^{N} \left(\frac{f^{(n+1)}(\xi)}{n!}(x - \xi)^n - \frac{f^{(n)}(\xi)}{(n-1)!}(x - \xi)^{n-1} \right) - \frac{R}{N!}(x - \xi)^N \\ &= \frac{f^{(N+1)}(\xi)}{N!}(x - \xi)^N - \frac{R}{N!}(x - \xi)^N \\ &= \frac{(x - \xi)^N}{N!}(f^{(N+1)}(\xi) - R) \end{aligned}$$

Clearly $F(x_0) = F(x) = f(x)$. So, by Rolle's Theorem, there is a point ξ between x_0 and x such that $F'(\xi) = 0$. At that point ξ we have $R = f^{(N+1)}(\xi)$ and thus we have

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}}{n!}(x - x_0)^n + \frac{f^{(N+1)}(\xi)}{(N+1)!}(x - x_0)^{N+1}.$$

This is what was to be proved. $\square$