UNIFORM CONTINUITY

JAMES KEESLING

The purpose of this reference is to give a brief synopsis of uniform continuity and an application. We begin with some definitions.

Let X be a metric space with metric d. A subset $A \subset X$ is said to be **compact** provided that every every sequence $\{x_i\}_{i=1}^{\infty}$ in A, there is a subsequence $\{x_{i_j}\}_{j=1}^{\infty}$ such that there is a $z \in A$ with

$$\lim_{j \to \infty} x_{i_j} = z.$$

Let $f : X \to Y$ be a function with Y a metric space with metric ρ. We say that f is **continuous at a point** $x_0 \in X$ provided that for every $\epsilon > 0$ there is a $\delta > 0$ such that for every $x \in X$, if $d(x, x_0) < \delta$, then $\rho(f(x), f(x_0)) < \epsilon$. A function $f : X \to Y$ is said to be **continuous** provided that it is continuous at each point $x \in X$.

A function $f : X \to Y$ is said to be **uniformly continuous** provided that for every $\epsilon > 0$, there is a $\delta > 0$ such that for every pair of points x and x' in X, with $d(x, x') < \delta$, $\rho(f(x), f(x')) < \epsilon$.

For a function $f : X \to Y$ to be uniformly continuous is stronger than being continuous. The main theorem is the following.

Theorem. Suppose that X is compact and that $f : X \to Y$ is continuous. Then f is uniformly continuous.

Proof. Suppose that $f : X \to Y$ is continuous, but not uniformly continuous. Then there is an $\epsilon_0 > 0$ such that for every $\delta > 0$, there are a pair of points a_δ and b_δ in X such that $d(a_\delta, b_\delta) < \delta$ with $\rho(f(a_\delta), f(b_\delta)) \geq \epsilon$. For this ϵ_0 and for each $i \geq 1$, choose a_i and b_i such that $d(a_i, b_i) < \frac{1}{i}$ with $\rho(f(a_i), f(b_i)) \geq \epsilon_0$.

By the compactness of X, there is a subsequence a_{ij} such that $a_{ij} \to z$ as $j \to \infty$ for some $z \in X$. By continuity of f at z, there is a $\delta > 0$ such that for all $x \in X$ with $d(x, z) < \delta$, $\rho(f(x), f(z)) < \frac{\epsilon_0}{2}$.

Now there is an j_0 such that $\frac{1}{i_{j_0}} < \frac{\delta}{2}$ and $d(a_{i_{j_0}}, z) < \frac{\delta}{2}$. Thus we have that $d(a_{i_{j_0}}, b_{i_{j_0}}) < \delta$. It is also true that $d(f(a_{i_{j_0}}), f(b_{i_{j_0}})) < \epsilon$. This is a contradiction. The contradiction implies that f must have been uniformly continuous, contrary to our assumption. The proof is complete. $\square$