

Section 4.7 — Applied Optimization

2. Wire of length 12 m is divided into two pieces and each piece is bent into a square. How should this be done in order to minimize the sum of the areas of the two squares.

(a) Express the areas in terms of the lengths x and y of the two pieces:

If I have a piece of wire of length x and I bend it into a square, each side is of length $\frac{x}{4}$, so the areas of the squares are $\frac{x^2}{16}$, and $\frac{y^2}{16}$.

(b) We are constrained by how much wire we have, so $x + y = 12$.

(c) The "interval of optimization" is just the range of numbers for which this problem makes any sense. The smallest x can be is 0, and the biggest it can be is 12. 0 because I can't use negative wire (what would that even mean?) and 12 because I only have 12 m of wire. I *can* use 0 or 12, though so my interval is $[0, 12]$ and it is closed.

(d) Yada yada now we solve it. We need to have a way to relate x and y and a formula to plug that into to optimize. Fortunately we already have both of these.

$$x + y = 12 \Rightarrow y = 12 - x$$

$$A = \frac{x^2}{16} + \frac{y^2}{16} = \frac{x^2}{16} + \frac{(12 - x)^2}{16}$$

$$A' = \frac{2x}{16} - \frac{2(12 - x)}{16}$$

$$0 = x - (12 - x) = 2x - 12$$

$$x = 6$$

So we have a critical point at $x = 6$. Looking at my original function, it is clearly going to have a positive x^2 term, which implies that the critical point is a *minimum*, which is what we want. We could also plug numbers into the first derivative to check. So $x = 6$, $y = 12 - x = 6$, so we should split the wire in half. Quick gut check: this gives us two squares with side length 1.5 m, this gives $(1.5)^2 + (1.5)^2 = 4.5m^2$, where as just not cutting the wire would give one square of size $3^2 = 9m^2$, so yeah, it makes sense that this would be a minimum.

4. Find the positive number x such that the sum of x and its reciprocal is as small as possible. Does this problem require optimization over an open interval or a closed interval.

So we want $x + \frac{1}{x}$ to be small. This is an open interval because $x > 0$, since we can't have $\frac{1}{0}$. Take the derivative:

$$f(x) = x + \frac{1}{x} \quad f'(x) = 1 - \frac{1}{x^2}$$

$$0 = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2}$$

So critical points are $x = 0, 1, -1$, but we can ignore 0 and -1 . Check signs: $f'(.5) = 1 - 4 = -3$, $f'(2) = 1 - .25 = .75$, so it's negative positive so this is a minimum as desired. So our number is $x = 1$.

5. A flexible tube of length 4m (I will choose to believe this is a pool noodle) is bent into an L-shape. Where should the bend be made to minimize the distance between the two ends. (Note: a good guess for this kind of thing is right in the middle. THIS DOES NOT ALWAYS WORK.)

So we get this L shape, one leg is x , the other is y , and we have that $x + y = 4\text{m}$, so $y = 4 - x$. The distance between the ends is the hypotenuse of this right triangle, so $d^2 = x^2 + y^2$, so $d^2 = x^2 + (4 - x)^2$.

$$d^2 = x^2 + 16 - 8x + x^2 = 2x^2 - 8x + 16$$

$$d'(2d) = 4x - 8$$

Okay, so in class Dr. Huang dealt with just the d' and ignored the $2d$. Why can we do this? Because the distance is always positive, and cannot result in a critical point. So the only way d' can be 0 is if $4x - 8 = 0, x = 2$. The original equation is a parabola pointing up, so $x = 2$ is a minimum. So the bend happens 2m from the end. (And look, it's the middle.)

6. Find the dimensions of the box with square base:

- (a) $V=12$, minimal surface area. $V = xyz$ where those are the length, width, and height, but $x = y$. So $V = 12 = x^2z$. Surface Area is: $S.A. = 2xy + 2xz + 2yz = 2x^2 + 4xz$. So $z = \frac{12}{x^2}$, which means

$$S.A. = 2x^2 + 4x \left(\frac{12}{x^2} \right) = 2x^2 + \frac{48}{x}$$

$$S.A.' = 4x - \frac{48}{x^2} = 0$$

$$\frac{4x^3 - 48}{x^2} = 0 \Rightarrow x^3 = 12, x = \sqrt[3]{12}, x = 0$$

Checking signs, the cube root is a min and 0 is a max. It asks for surface area, so we have

$$2(12)^{\frac{2}{3}} + 48(12)^{-\frac{1}{3}} = 2(12)^{2/3} + \frac{48(12)^{\frac{2}{3}}}{12} = 6(12)^{\frac{2}{3}}$$

- (b) $S.A. = 20$, maximize volume. $20 = 2x^2 + 4xz, z = \frac{20-2x^2}{4x}$

$$V = x^2z = x^2 \left(\frac{20 - 2x^2}{4x} \right) = 20x - 2x^3$$

$$V' = 20 - 6x^2, \Rightarrow x = \sqrt{\frac{10}{3}}$$

And checking shows this is a max.

$$V = \frac{10}{3} \frac{20 - 2\frac{10}{3}}{\sqrt{\frac{10}{3}}} = \frac{10\sqrt{30}}{9}$$

9. So the dimensions of the rectangle are $x = 2(4) \cos \theta, y = 2(4) \sin \theta$.

$$V = 4(16) \sin \theta \cos \theta = 48 \sin \theta \cos \theta = 24 \sin 2\theta$$

$$V' = 48 \cos 2\theta = 0 \Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$$

So the angle is 45 degrees, so $x = y = 4\sqrt{2}$. (notice that again this was right in the middle.)

26. Find the maximum area of a triangle formed by the axes and a tangent line to the graph of $y = (x + 1)^{-2}$ with $x > 0$.

So we need to start by writing the equation of a tangent line:

$$y - f(a) = f'(a)(x - a), \quad y' = \frac{-2}{(x + 1)^3}$$

$$y - \frac{1}{(a + 1)^2} = \frac{-2}{(a + 1)^3}(x - a)$$

$$y = \frac{1}{(a + 1)^2} + \frac{-2(x - a)}{(a + 1)^3}$$

Now we need to plug in $x = 0$ to get the y -intercept and $y = 0$ for the x -intercept. Some thought and a sketch tells us that the base of the triangle has length equal to the x -intercept and height equal to the y -intercept.

$$0 = \frac{1}{(a + 1)^2} + \frac{-2(x - a)}{(a + 1)^3}$$

$$-\frac{1}{(a + 1)^2} = \frac{-2(x - a)}{(a + 1)^3}$$

$$-(a + 1) = -2(x - a)$$

$$a + 1 = 2x - 2a$$

$$x = \frac{3a + 1}{2}$$

$$y = \frac{1}{(a + 1)^2} + \frac{-2(0 - a)}{(a + 1)^3}$$

$$y = \frac{(a + 1) + 2a}{(a + 1)^3}$$

$$y = \frac{3a + 1}{(a + 1)^3}$$

So the area of the triangle is $\frac{1}{2}bh$

$$A = \frac{1}{2}bh = \frac{1}{2} \frac{3a + 1}{2} \frac{3a + 1}{(a + 1)^3} = \frac{(3a + 1)^2}{4(a + 1)^3}$$

$$A' = \frac{1}{4} \frac{(a + 1)^3(3 * 2(3a + 1)) - (3a + 1)^2(3(a + 1)^2)}{(a + 1)^6} = \frac{3(a + 1)^2(3a + 1)(2(a + 1) - (3a + 1))}{4(a + 1)^6}$$

$$A' = \frac{3(a + 1)^2(3a + 1)(2a + 2 - 3a - 1)}{4(a + 1)^6} = \frac{3(a + 1)^2(3a + 1)(1 - a)}{4(a + 1)^6}$$

And this gives only 1 zero which is positive: $a = 1$. All that work for that. So we plug this into our two intercepts:

$$x = \frac{3(1) + 1}{2} = 2, y = \frac{3(1) + 1}{(1 + 1)^3} = \frac{4}{2^3} = \frac{1}{2}$$

$$A = \frac{1}{2}(2) \left(\frac{1}{2}\right) = \frac{1}{2}$$

39. So the equation we're going to use is going to be based around how many times they jack up the rent. If they don't change the rent, we have 100 rooms rented out, and the price is 900. If they jack up the rent n times, our rent is $900 + 10n$ and we will have $100 - n$ rooms rented out. So if we stop and think about it, that says that if they increase the rent to 970, we will have $100 - 7 = 93$ rooms rented out, so that makes sense. Now we want an equation that expresses how much money we make. Also note that for each room, we don't actually get the rent, we get the rent *minus* the maintenance cost. So our profit is

$$\begin{aligned} P(n) &= (100 - n)(900 + 10n - 80) = (100 - n)(820 + 10n) = 10(100 - n)(82 + n) \\ &= 10(82000 + 100n - 82n - n^2) = 10(82000 + 18n - n^2) \\ P'(n) &= 10(18 - 2n) \Rightarrow n = 9 \end{aligned}$$

And it asked for the *rent*, so the rent is $900 + 10n$, so we have $900 + 90 = \$990$.