Vertical Paths in Simple Varieties of Trees

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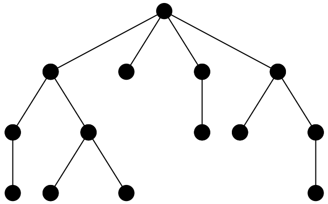
Simple Varieties of Trees

A graph $G = (V, E)$ consists of a set of vertices V and a set of edges E.

Definition

A rooted tree is a connected graph without cycles with one vertex distinguished as the root. A class of trees is called *simple variety* if the generating function for the number of trees on n vertices satisfies an equation of the form

$$T(x) = x\phi(T(x)).$$



Motivation

Things modelled by rooted trees:

- Computer network access
- Social networks
- Distribution networks

Rooted trees are also used as data structures.

A Few Simple Varieties of Trees

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- Binary trees: unlabeled, plane, children are designated left or right, 0, 1, or 2 children. $T(x) = x(1 + 2T(x) + T(x)^2)$.

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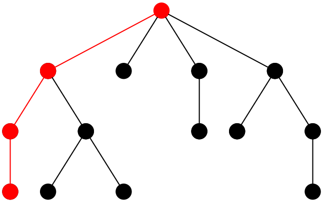
- Binary trees: unlabeled, plane, children are designated left or right, 0, 1, or 2 children. $T(x) = x(1 + 2T(x) + T(x)^2)$.
- Cayley trees: labeled, nonplane, any number of children. $T(x) = xe^{T(x)}$.

Some Traits of Trees/Vertices

- The *height* of a tree is the distance from the root to the furthest vertex plus one.



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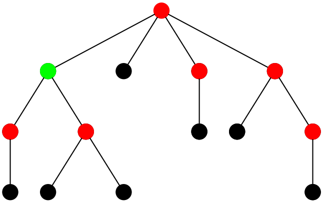
- The *rank/protection number* of a vertex is the shortest distance from that vertex to its closest descendant leaf.



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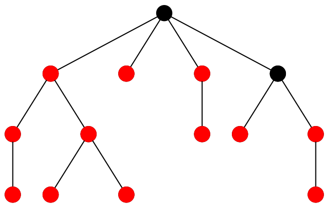
- The *height* of a tree is the distance from the root to the furthest vertex plus one.
- The *rank/protection number* of a vertex is the shortest distance from that vertex to its closest descendant leaf.
- A tree or vertex is *balanced* if its height is one greater than its rank.



Previous Results

Theorem (Flajolet and Odlyzko, 1981)

The expected height of any simple variety of tree is asymptotically $k\sqrt{\pi n}$ for some $k > 0$.

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Theorem (Flajolet and Sedgewick, 2009)

The sum of the lengths of paths where the root is one endpoint is of order $n^{3/2}$.

Previous Results

Theorem (C, 2016)

The expected rank of the root of a uniformly chosen general tree approaches

$$\sum_{k=1}^{\infty} \frac{9}{4^{1-k} + 4 + 4^k} \approx 1.62297.$$

The expected rank of a uniformly chosen vertex in a uniformly chosen general tree approaches

$$\sum_{k=1}^{\infty} \frac{3}{4^k + 2} \approx 0.727649.$$

Counting Leaves

Proposition

Let $T(x)$ be the generating function for the number of trees on n vertices in some simple variety of trees, and let $L(x)$ be the generating function for the number of trees with a marked leaf in the same family. Then

$$L(x) = \frac{x^2 T'(x)}{T(x)}.$$

It suffices to show that there is a bijection between trees with a marked vertex and pairs of trees and trees with a marked leaf with one vertex removed. This would show that

$$xT'(x) = V(x) = \frac{T(x)L(x)}{x}.$$

Counting Leaves

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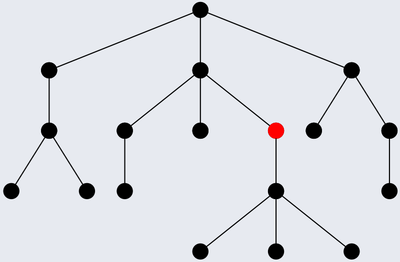
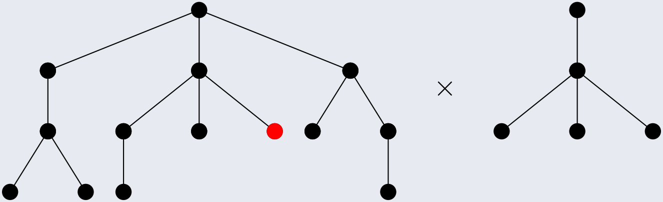
Proof.





□

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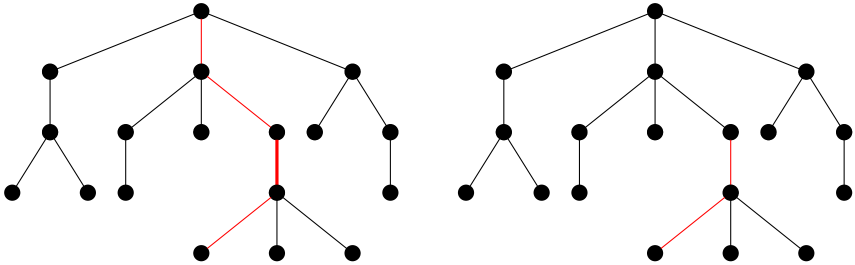
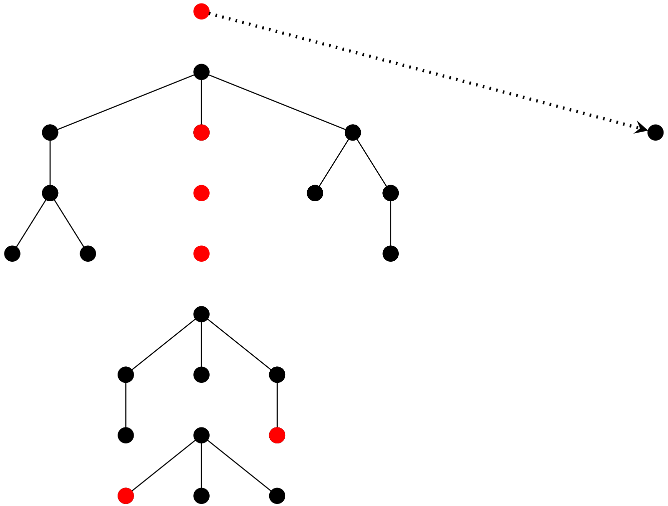
- Any vertical path: $\frac{L(x)}{x}(V(x) - T(x))$
- Vertical paths that end in a leaf: $\frac{L(x)}{x}(L(x) - x)$
- Edges in paths from the root: $\frac{L(x)}{x}(V(x) - T(x))$



Higher Moments

Consider the generating function $\left(\frac{L(x)}{x}\right)^k (V(x) - T(x))$.



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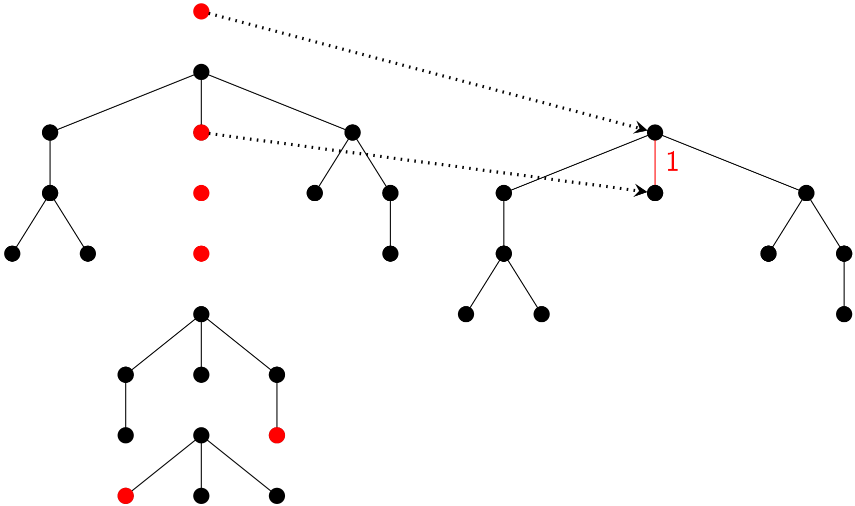

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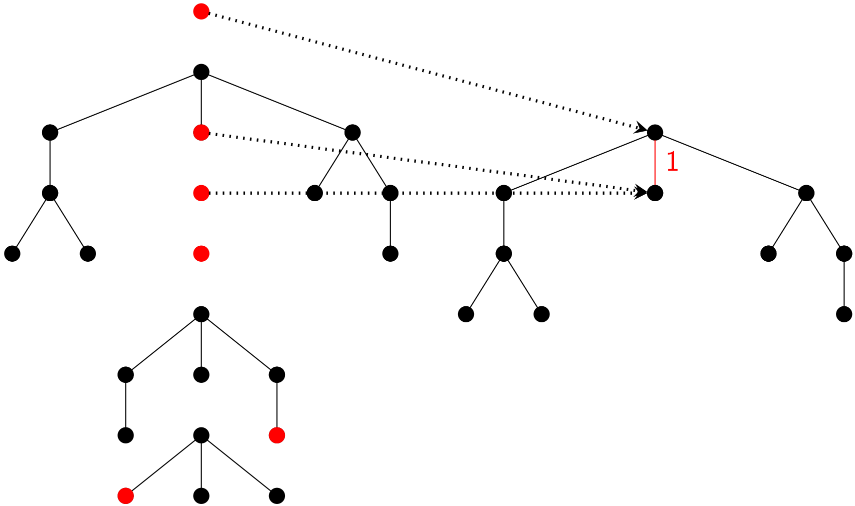

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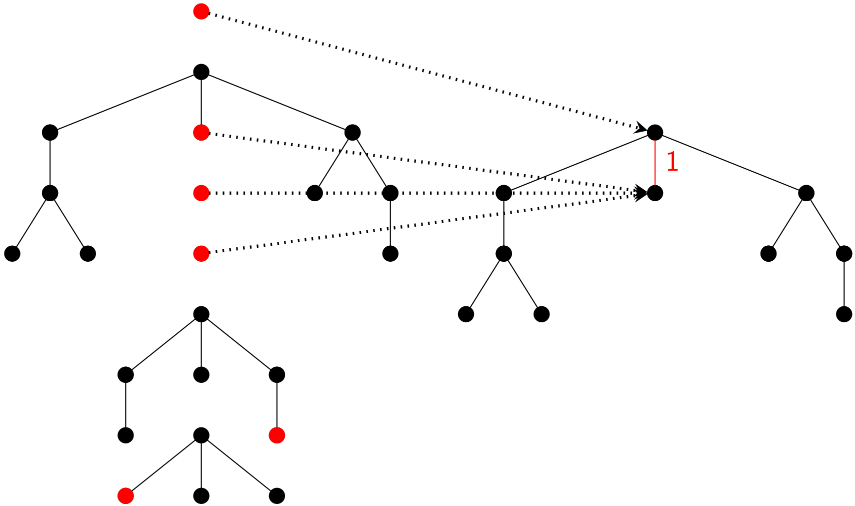

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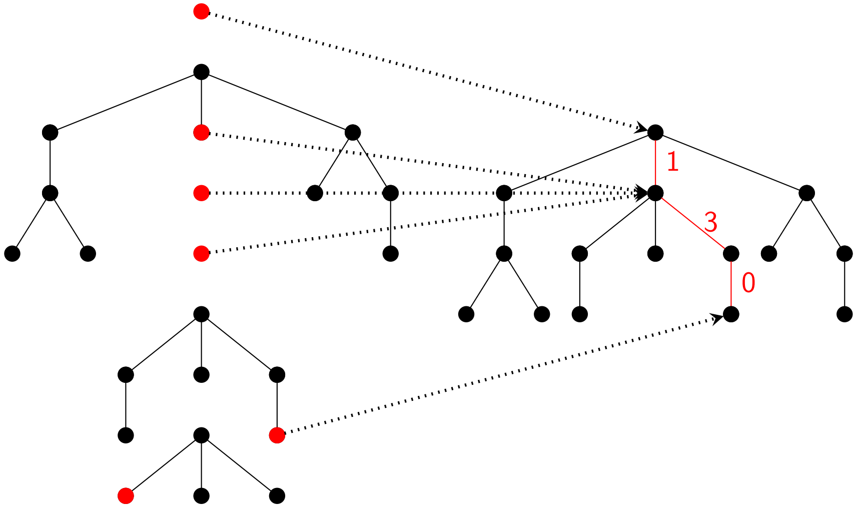

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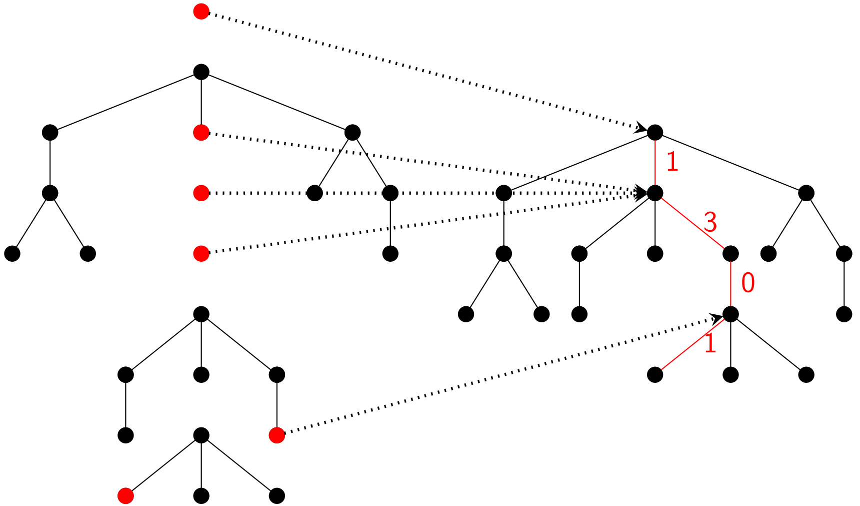

Higher Moments

Proposition

The generating function $\left(\frac{L(x)}{x}\right)^k (V(x) - T(x))$ *counts the number of paths in all trees of some variety on* n *vertices, with each path weighted by the number of* k *element multi-sets of the edges.*

Proposition

The generating function for the number of paths from the root in all trees of some variety on n *vertices, with each path weighted by the length of the path to the* k*th power is a polynomial of degree* k *of the form*

$$\left(k!\left(\frac{L(x)}{x}\right)^k - \frac{k!(k-1)}{2}\left(\frac{L(x)}{x}\right)^{k-1} + Q_k\left(\frac{L(x)}{x}\right)\right)(V(x) - T(x)),$$

where $Q_k(x)$ *is a polynomial in* $\frac{L(x)}{x}$ *of degree* $k-2$ *or less.*

Higher Moments

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Proof.

Let the length of a given path be ℓ. Then in the generating function $\left(\frac{L(x)}{x}\right)^k (V(x) - T(x))$ has weight $\binom{\ell+k-1}{k} = \frac{1}{k!}\left(\ell^k - \frac{k(k-1)}{2}\ell^{k-1} + \dots\right)$, this gives the leading coefficient. Proceeding by induction, we can remove the lower order terms by subtracting lower order polynomials.

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Asymptotics

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In any simple variety of tree,

$$T(x) = a_0 - a_1\sqrt{1-\frac{x}{\rho}} + a_2\left(1-\frac{x}{\rho}\right) + O\left(\left(1-\frac{x}{\rho}\right)^{3/2}\right),$$

with $a_0, a_1 > 0$.

$$V(x) = \frac{a_1}{2\sqrt{1-\frac{x}{\rho}}} - a_2 + O\left(\sqrt{1-\frac{x}{\rho}}\right),$$

$$\frac{L(x)}{x} = \frac{a_1}{2a_0\sqrt{1-\frac{x}{\rho}}} + \frac{\left(a_1^2 - 2a_0a_2\right)}{2a_0^2} + O\left(\sqrt{1-\frac{x}{\rho}}\right).$$

Asymptotics

Let $X(n)$ be the r.v. whose value is the length of a uniformly randomly selected path from the root of a tree on n vertices to any vertex. Then we can compute

$$\mathbb{E}[X(n)^k] = n^{k/2}\left(\left(\frac{a_1}{a_0}\right)^k \Gamma\left(\frac{k}{2}+1\right) - \frac{a_1^{k-1}\left((k+1)a_0^2 + 2a_2(k+1)a_0 - a_1^2 k\right) k(k-1)\Gamma\left(\frac{k-1}{2}\right)}{4a_0^{k+1}\sqrt{n}}\right) + O\left(n^{k/2-1}\right),$$

as well as corresponding expectations for three other variations of vertical paths.

Asymptotics

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In the family of Cayley trees $a_0 = 1, a_1 = \sqrt{2}, a_2 = \frac{2}{3}$. The expected length of a path from

- the root to any vertex: $\sqrt{\frac{\pi n}{2}} - \frac{4}{3}$.
- the root to any leaf: $\sqrt{\frac{\pi n}{2}} - \frac{1}{3}$.
- any vertex to any of its descendants: $\sqrt{\frac{2n}{\pi}} + \frac{8}{3\pi} - 1$.
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- any vertex to any of its descendant leaves: $\sqrt{\frac{2n}{\pi} + \frac{2}{3\pi}}$.

These trends are universal, if done with general terms, fixing a type of top point, leaves are a constant further away ($\frac{a_1^2}{2a_0^2}$ with roots, $\frac{a_1^2(\pi-2)}{2a_0^2(\pi)}$ with vertices) plus an error term, fixing a type of bottom point, roots are further by a factor of $\frac{\pi}{2}$ plus an error term (which depends on the type of tree).

Related Problems

- Expected length of paths from the root to a leaf in various families
- Bijective proofs of correspondence between all paths and vertical paths