Cor: If (X,A) Has HEP And $A \simeq *$, Then $q: X \to X/A$ Is A Homotopy Equiv.

Pf: Let $f_t: X \to X$ Be A Homotopy Extending A Contraction Of A With $f_0 = \mathrm{Id}_X$ And $f_t(A) \subset A$ For All t. The Composition $q f_t : X \to X/A$ Sends A To A Point And So Factors As A Composition

$$\begin{tikzcd} X \arrow[r, "q"] \arrow[dr, "qf_t"'] & X/A \arrow[d, dashed, "\exists!\, \bar{f}_t"] \\ & X/A \end{tikzcd}$$

(UMP Of Quotients)

When $t=1$, $f_1(A)$ Is A Point. So f_1 Induces A Map $g: X/A \to X$ With $gq = f_1$ (UMP)

But Then We Have $qg = \bar{f}_1$ Since $qg(\bar{x}) = qgq(x) = q f_1(x) = \bar{f}_1 q(x) = \bar{f}_1(\bar{x})$

$$\begin{tikzcd} X \arrow[r, "f_1"] \arrow[d, "q"'] & X \arrow[d, "q"] \\ X/A \arrow[ur, "\exists!\, g"] \arrow[r, "\bar{f}_1"'] & X/A \end{tikzcd}$$

So g And q Are Inverse Homotopy Equivalences:

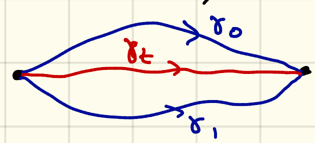
$gq = f_1 \simeq f_0 = \mathrm{Id}_X$

$qg = \bar{f}_1 \simeq \bar{f}_0 = \mathrm{Id}_{X/A}$. //

The Fundamental Group

Read Intro To Chapter 1 Of Hatcher For A Nice Motivation

Def: A Path In X Is A Continuous Map $\gamma: [0,1] \to X$. Two Paths γ_0, γ_1 Are Homotopic (rel $\{0,1\}$) If (1) $\gamma_0(0) = \gamma_1(0)$; (2) $\gamma_0(1) = \gamma_1(1)$; And (3) There Is A Homotopy $F: I \times I \to X$ With $F(s,0) = \gamma_0(s)$, $F(s,1) = \gamma_1(s)$, $F(0,t) = \gamma_0(0)$, $F(1,t) = \gamma_0(1)$. We Write $\gamma_0 \simeq \gamma_1$, And $\gamma_t = F(-,t)$.



Prop: Homotopy rel $\{0,1\}$ Is An Equivalence Relation On The Set Of Paths $\{\gamma: I \to X\}$.

Pf: 1. $\gamma_0 \simeq \gamma_0$ Is Clear.

2. If $\gamma_0 \simeq \gamma_1$ Via $F: I \times I \to X$, Then $\gamma_1 \simeq \gamma_0$ Via $G: I \times I \to X$ Defined By $G(s,t) = F(s, 1-t)$.

3. If $\gamma_0 \underset{F}{\simeq} \gamma_1$ And $\gamma_1 \underset{G}{\simeq} \gamma_2$ Define $H: I \times I \to X$ By $H(s,t) = \begin{cases} F(s,2t), & 0 \le t \le \frac{1}{2} \\ G(s,2t-1), & \frac{1}{2} \le t \le 1 \end{cases}$. Then $\gamma_0 \underset{H}{\simeq} \gamma_2$. //

Composition Of Paths

Two Paths α And β Are Composable If $\alpha(1) = \beta(0)$. Define The Product Path $\alpha \cdot \beta$ By

$$\alpha \cdot \beta(s) = \begin{cases} \alpha(2s) & 0 \le s \le \frac{1}{2} \\ \beta(2s-1) & \frac{1}{2} \le s \le 1. \end{cases}$$

Note: If $\alpha_0 \simeq \alpha_1$ And $\beta_0 \simeq \beta_1$ Then $\alpha_0 \cdot \beta_0 \simeq \alpha_1 \cdot \beta_1$. Indeed, If α_t And β_t Are Homotopies Then $\alpha_t \cdot \beta_t$ Is A Homotopy From $\alpha_0 \cdot \beta_0$ To $\alpha_1 \cdot \beta_1$.

Now, Consider A Point $x_0 \in X$ And The Set Of Loops Based At x_0: $\gamma: I \to X$, $\gamma(0) = \gamma(1) = x_0$.

All Loops At x_0 Are Composable.

Def: $\pi_1(X,x_0)$ Is The Set Of Homotopy Classes rel $\{0,1\}$ Of Loops Based At x_0.

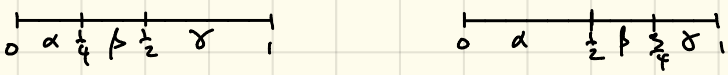
Prop: $\pi_1(X,x_0)$ Is A Group Under Path Composition.

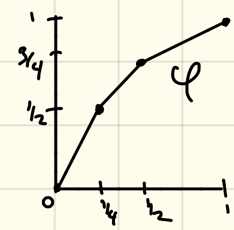
Pf: Denote The Homotopy Class Of α By $[\alpha]$. Since Composition Respects Homotopy, The Operation $[\alpha]\cdot[\beta] = [\alpha\cdot\beta]$ Is Well-Defined. Need To Check The Group Axioms.

First: A Reparametrization Of γ Is A Composition $\gamma\circ\varphi$, Where $\varphi: I \to I$ Is A Map With $\varphi(0)=0$ And $\varphi(1)=1$. Any Such φ Is Homotopic To id_I Via $\Phi(s,t) = (1-t)\varphi(s)+ts$. So $\gamma\circ\varphi \simeq \gamma$ For Any φ.

Note That Path Composition Is Not Associative In General:

$(\alpha\cdot\beta)\cdot\gamma$ $\qquad\qquad$ $\alpha\cdot(\beta\cdot\gamma)$

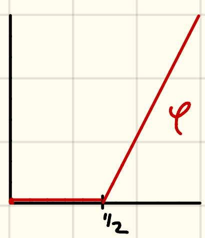




But: $(\alpha\cdot\beta)\cdot\gamma$ & $\alpha\cdot(\beta\cdot\gamma)$ Differ By A Reparametrization

Thus, $[\alpha]\cdot([\beta]\cdot[\gamma]) = ([\alpha]\cdot[\beta])\cdot[\gamma]$.

Identity: Define $\eta_{x_0}: I \to X$ By $\eta_{x_0}(s) = x_0$ For All $s \in I$. Then $\eta_{x_0}\cdot\gamma \simeq \gamma\cdot\eta_{x_0} \simeq \gamma$:



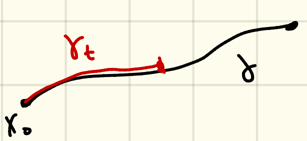
$\gamma\circ\varphi = \eta_{x_0}\cdot\gamma$

$\gamma\circ\psi = \gamma\cdot\eta_{x_0}$

So $[\eta_{x_0}]$ Is A 2-Sided Identity.

Inverse: If $\gamma: I \to X$ Is A Path From x_0 To x_1, Then $\gamma^{-1}: I \to X$ Defined By $\gamma^{-1}(s) = \gamma(1-s)$ Is A Path From x_1 To x_0. Note That $\gamma\cdot\gamma^{-1} \simeq \eta_{\gamma(0)}$ And $\gamma^{-1}\cdot\gamma \simeq \eta_{\gamma(1)}$

To See This, Let $\gamma_t = \gamma$ On $[0,1-t]$, Stationary At $\gamma(1-t)$ On $[1-t,1]$

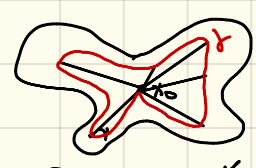


Let $h_t = \gamma_t\cdot\gamma_t^{-1}$. Then This Is A Homotopy From $\gamma\cdot\gamma^{-1}$ To $\eta_{\gamma(0)}$

So If We Have A Loop α At x_0, Then $\alpha\cdot\alpha^{-1} \simeq \eta_{x_0}$ And So $[\alpha]\cdot[\alpha^{-1}] = [\eta_{x_0}]$ And $[\alpha^{-1}]\cdot[\alpha] = [\eta_{x_0}]$. //

Def: A Space X Is Simply Connected If $\pi_1(X,x_0) = \{[\eta_{x_0}]\}$.

eg: Suppose X Is A Star-Shaped Subset Of $\mathbb{R}^n$; i.e. There Is $x_0 \in X$ Such That The Line Segment Joining x_0 To y Lies In X For Every $y \in X$.

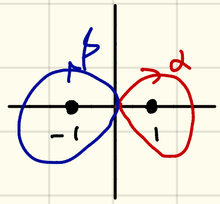


Claim: $\pi_1(X,x_0) = \{[\eta_{x_0}]\}$.

Pf: If $y \in X$, Let l_y Be The Line Segment From x_0 To y, Parametrized As $l_y(t) = (1-t)x_0 + ty$. If γ Is A Loop At x_0, Define $F: I \times I \to X$ By $F(s,t) = l_{\gamma(s)}(t)$. Then F Is Continuous And $F(s,0) = l_{\gamma(s)}(0) = x_0$, $F(s,1) = l_{\gamma(s)}(1) = \gamma(s)$

$F(0,t) = l_{\gamma(0)}(t) = l_{x_0}(t) = x_0 \Rightarrow F: \gamma \simeq \eta_{x_0}$.

eg: $X = \mathbb{C} - \{\pm 1\}$



$\Theta : \pi_1(X,0) \to F_2$ = FREE GROUP ON $\{a,b\}$
$\alpha \mapsto a$
$\beta \mapsto b$
IS SURJECTIVE (IN FACT AN ISOMORPHISM)

THE INDUCED HOMOMORPHISM

SUPPOSE $f : (X,x_0) \to (Y,y_0)$ IS CONTINUOUS. DEFINE $f_* : \pi_1(X,x_0) \to \pi_1(Y,y_0)$ BY $f_*([\alpha]) = [f \circ \alpha]$. THIS IS WELL-DEFINED: IF $\alpha_0 \simeq \alpha_1$ rel$\{0,1\}$, THEN $f \circ \alpha_0 \simeq f \circ \alpha_1$ rel$\{0,1\}$ (Pf: IF $F : I \times I \to X$ IS A HOMOTOPY rel$\{0,1\}$ $\alpha_0 \simeq \alpha_1$, THEN $f \circ F : I \times I \to Y$ IS A HOMOTOPY rel$\{0,1\}$ FOR $f \circ \alpha_0 \simeq f \circ \alpha_1$.)

f_* IS A HOMOMORPHISM: IF α, β COMPOSABLE IN X, THEN $f \circ \alpha$ AND $f \circ \beta$ ARE COMPOSABLE IN Y AND $f \circ (\alpha \cdot \beta) = (f \circ \alpha) \cdot (f \circ \beta)$ (WRITE DOWN THE DEFINING FORMULAS TO CHECK).

THIS PLAYS NICELY WITH RESPECT TO COMPOSITION:

$id_* : \pi_1(X,x_0) \to \pi_1(X,x_0)$ IS THE IDENTITY

$$\begin{tikzcd} \pi_1(X,x_0) \arrow[r, "f_*"] \arrow[dr, "(g\circ f)_*"'] & \pi_1(Y,y_0) \arrow[d, "g_*"] \\ & \pi_1(Z,z_0) \end{tikzcd}$$

$(g \circ f)_* = g_* \circ f_*$

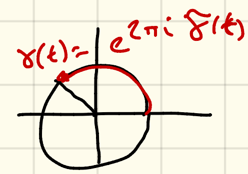
IN CATEGORICAL LANGUAGE: π_1 IS A FUNCTOR FROM $\{$POINTED SPACES$\} \to \{$GROUPS$\}$

THM: $\pi_1(S^1, 1) \cong \mathbb{Z}$

PROOF: VIEW $S^1 \subset \mathbb{C}$ AS $\{z \mid |z| = 1\}$. I CLAIM THAT EVERY CONTINUOUS $\gamma : ([0,1],\{0,1\}) \to (S^1, 1)$ LIFTS TO A CONTINUOUS $\tilde{\gamma} : ([0,1], 0) \to (\mathbb{R}, 0)$.

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "\exp"] \\ {[0,1]} \arrow[ur, "\tilde{\gamma}"] \arrow[r, "\gamma"'] & S^1 \end{tikzcd}$$

INSTEAD, WE CAN WRITE $\gamma(t) = \exp(2\pi i \tilde{\gamma}(t))$ WHERE $\tilde{\gamma} : [0,1] \to \mathbb{R}$. NOTE THAT $\tilde{\gamma}(1) \in \mathbb{Z}$ SINCE $\gamma(1) = 1$. WE WILL SHOW THAT THE "WINDING NUMBER MAP"



$\tilde{\gamma}$ = ANGLE

$\pi_1(S^1,1) \to \mathbb{Z}$ $[\gamma] \mapsto \tilde{\gamma}(1)$ IS (a) WELL-DEFINED, (b) A HOMOMORPHISM, (c) BIJECTIVE.

NOW, SUPPOSE $\gamma, \mu : ([0,1],\{0,1\}) \to (S^1, 1)$. THEN $\tilde{\gamma}(1) = \tilde{\mu}(1) \iff \gamma \simeq \mu$ rel$\{0,1\}$

Pf: ($\Rightarrow$) SUPPOSE $\tilde{\gamma}(1) = \tilde{\mu}(1) = n$. DEFINE $\Phi(s,t) = s\tilde{\gamma}(t) + (1-s)\tilde{\mu}(t)$. THEN $\Phi(s,0) = 0$ FOR ALL s, + $\Phi(s,1) = n$ FOR ALL s + SO $\Phi : \tilde{\gamma} \simeq \tilde{\mu}$. DEFINE $F(s,t) = e^{2\pi i \Phi(s,t)}$ THEN $F : \gamma \simeq \mu$ rel$\{0,1\}$.

($\Leftarrow$) IF $F : \gamma \simeq \mu$ rel$\{0,1\}$, THEN WE CAN LIFT THE HOMOTOPY: $\tilde{F} : I \times I \to \mathbb{R}$, $\tilde{F}(s,t) = \tilde{\gamma}_t(s)$

Then $\tilde F(s,0)=\tilde\gamma_0(s)$, $\tilde F(s,1)=\tilde\mu(s)$, $\tilde F(0,t)=\tilde\gamma_t(0)=0$, $\tilde F(1,t)=\tilde\gamma_t(1)=\tilde\gamma(1)\ \forall t$
$=\tilde\mu(1)$ since $\tilde\gamma_1=\tilde\mu$

This Proves That The Map Is Well-Defined And Injective. For Surjectivity, Note That If We Set $\omega_n(t)=e^{2\pi i n t}$, Then $\tilde\omega_n(t)=nt$ And $\tilde\omega_n(1)=n$. This Maps To $n\in\mathbb{Z}$. The Map Is A Homomorphism:

i.e. $\widetilde{\gamma\cdot\mu}(1)=n+m$. //

Products

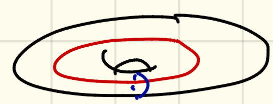
Suppose (X,x_0) And (Y,y_0) Are Pointed Spaces.

$$\begin{array}{c} (X,x_0)\\ \uparrow p\\ (X\times Y,(x_0,y_0))\\ \downarrow q\\ (Y,y_0)\end{array} \quad\Longrightarrow\quad \begin{array}{ccc} \pi_1(X,x_0) & \xrightarrow{i_X} & \\ \uparrow p_* & & \\ \pi_1(X\times Y,(x_0,y_0)) & \longrightarrow & \pi_1(X,x_0)\times\pi_1(Y,y_0)\\ \downarrow q_* & & \\ \pi_1(Y,y_0) & \xrightarrow{i_Y} & \end{array}$$

Have A Homomorphism $p_*\times q_*:\pi_1(X\times Y,(x_0,y_0))\longrightarrow\pi_1(X,x_0)\times\pi_1(Y,y_0)$
$$[\gamma]\longmapsto(p_*[\gamma],q_*[\gamma])$$

Claim: This Map Is An Isomorphism.

eg: $T^n=(S^1)^n$ So $\pi_1(T^n,1)\cong\mathbb{Z}^n$. $n=2$: $\pi_1(T)=\mathbb{Z}\times\mathbb{Z}$
$[\gamma]=(m,n)$
m Times Around, n Times Around



Pf Of Claim: Products Have A Universal Mapping Property: $\varphi:Z\to X\times Y$ Is Continuous $\Leftrightarrow$ $p\circ\varphi:Z\to X$ + $q\circ\varphi:Z\to Y$ Are Continuous.

Surjectivity: Suppose $([\gamma],[\mu])\in\pi_1(X,x_0)\times\pi_1(Y,y_0)$. This Is Represented By A Pair Of Loops $\gamma:([0,1],\{0,1\})\to(X,x_0)$ + $\mu:([0,1],\{0,1\})\to(Y,y_0)$. By The UMP The Map $\gamma\times\mu:([0,1],\{0,1\})\to(X\times Y,(x_0,y_0))$ Is Continuous + $p_*\times q_*([\gamma\times\mu])=([\gamma],[\mu])$

Injectivity: Suppose $(p_*\times q_*)[\alpha]=(p_*\times q_*)[\beta]$ Where $\alpha,\beta:([0,1],\{0,1\})\to(X\times Y,(x_0,y_0))$

Let $\alpha_X=p\circ\alpha$, $\alpha_Y=q\circ\alpha$, $\beta_X=p\circ\beta$, $\beta_Y=q\circ\beta$. Then $(p_*\times q_*)[\alpha]=([\alpha_X],[\alpha_Y])$
$(p_*\times q_*)[\beta]=([\beta_X],[\beta_Y])$

By Assumption, $\alpha_X\simeq\beta_X$ rel $\{0,1\}$ And $\alpha_Y\simeq\beta_Y$ rel $\{0,1\}$. Let $F_X:I\times I\to X$ + $F_Y:I\times I\to Y$ Be These Homotopies. Let $F:I\times I\to X\times Y$ Be The Corresponding Continuous Map $F_X\times F_Y$. Then $F_X\times F_Y:\alpha\simeq\beta$ (Exercise)

Applications

1. Fundamental Thm of Algebra

Suppose $p(z) = z^n + a_1 z^{n-1} + \dots + a_n$ has no roots in $\mathbb{C}$. If $r \geq 0$ then

$$f_r(s) = \frac{p(re^{2\pi i s})/p(r)}{|p(re^{2\pi i s})/p(r)|}$$

defines a loop in S^1, based at 1. As r varies, f_r is a homotopy of loops in S^1 based at 1. Note that f_0 is the trivial loop & so $[f_r] \in \pi_1(S^1) \cong \mathbb{Z}$ is 0 for all $r \geq 0$.
Now, fix $r \gg 0$, bigger than $|a_1| + \dots + |a_n| + 1$. Then for $|z| = r$ we have

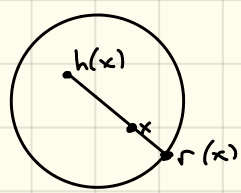
$$|z^n| > (|a_1| + \dots + |a_n|)\,|z^{n-1}| > |a_1 z^{n-1}| + \dots + |a_n|$$
$$\geq |a_1 z^{n-1} + \dots + a_n|$$

It follows that $p_t(z) = z^n + t(a_1 z^{n-1} + \dots + a_n)$ has no roots on the circle $|z| = r$ when $0 \leq t \leq 1$. Replacing p by p_t in the formula for f_r and letting t go from 1 to 0 we obtain a homotopy from f_r to $\omega_n(s) = e^{2\pi i n s}$. But $[\omega_n] = n \in \pi_1(S^1)$ and since $[\omega_n] = [f_r] = 0$, we have $n = 0$. That is, p is constant. //

2. Brouwer Fixed Point Thm for the Disc

Every continuous map $h: D^2 \to D^2$ has a fixed point.

Pf: Suppose not. Define $r: D^2 \to S^1$ by



Note that $r|_{S^1} = id_{S^1}$ and r is clearly continuous.
But then we have a commutative diagram

$$\begin{tikzcd}
\pi_1(S^1,1) \arrow[r, "i_*"] \arrow[dr, "id"'] & \pi_1(D^2,1) = 0 \arrow[d, "r_*"] \\
 & \pi_1(S^1,1)
\end{tikzcd}$$

So that $id: \pi_1(S^1) \to \pi_1(S^1)$ is the zero map, a contradiction. //

3. Borsuk-Ulam Thm in Dimension 2

If $f: S^2 \to \mathbb{R}^2$ is continuous then there is an $x \in S^2$ with $f(x) = f(-x)$.

Pf: Suppose not. Define $g: S^2 \to S^1$ by $g(x) = \{f(x) - f(-x)\}/|f(x) - f(-x)|$. Define a loop η on S^2 by $\eta(s) = (\cos 2\pi s, \sin 2\pi s, 0)$ and let $h = g \circ \eta$. Since $g(-x) = -g(x)$ we have $h(s + \frac{1}{2}) = -h(s)$ for all $s \in [0, \frac{1}{2}]$. Lift h to a map $\tilde{h}: I \to \mathbb{R}$. Since $h(s + \frac{1}{2}) = -h(s)$ we have $\tilde{h}(s + \frac{1}{2}) = \tilde{h}(s) + q/2$ for some odd integer q. q might depend on s, but it's easy to see that it doesn't as it must depend continuously on s and so must be constant since $q \in \mathbb{Z}$. In particular, $\tilde{h}(1) = \tilde{h}(\frac{1}{2}) + q/2 = \tilde{h}(0) + q$. So h represents q times a generator of $\pi_1(S^1)$ and since q is odd, h is not nullhomotopic. But η is obviously nullhomotopic in S^2 & so $g \circ \eta$ is nullhomotopic in S^1, a contradiction. //

Prop: $\pi_1(S^n) = 0 \quad n \geq 2$

Pf: First Prove The Following Lemma: If X Is The Union Of A Collection Of Path Connected Open Sets A_α Each Containing The Base Point $x_0 \in X$ And Each Intersection $A_\alpha \cap A_\beta$ Is Path Connected, Then Every Loop In X Is Homotopic To A Product Of Loops, Each Contained In A Single A_α.

Pf: Suppose $\gamma: I \to X$ Is A Loop At x_0. Since γ Is Continuous, Each $s \in I$ Has A Nbhd V_s In I Mapped By γ Into Some A_α; In Fact We May Assume $\bar{V}_s$ Maps To A_α. Since I Is Compact, A Finite Collection Of These V_s Cover I; Say $0 = s_0 < s_1 < \dots < s_m = 1$ Is The Corresponding Collection Of s Values. Denote The A_α Containing $\gamma([s_{i-1}, s_i])$ By A_i And Let γ_i Be The Path $\gamma|_{[s_{i-1}, s_i]}$. Then $\gamma = \gamma_1 \cdot \gamma_2 \cdots \gamma_m$ With γ_i In A_i. Since $A_i \cap A_{i+1}$ Is Path Connected, Choose g_i In $A_i \cap A_{i+1}$ From x_0 To $\gamma(s_i) \in A_i \cap A_{i+1}$. Then The Loop $(\gamma_1 \cdot g_1^{-1}) \cdot (g_1 \cdot \gamma_2 \cdot g_2^{-1}) \cdot \dots \cdot (g_{m-1} \cdot \gamma_m) \simeq \gamma$ And Each Piece Lies In Some A_i. //

Now Write $S^n = A_1 \cup A_2$, Where $A_1 = S^n - \{(0,0,\dots,1)\}$ And $A_2 = S^n - \{(0,0,\dots,0,-1)\}$. Then $A_1 \cong \mathbb{R}^n \cong A_2$ And $A_1 \cap A_2 \cong S^{n-1} \times \mathbb{R}$. Choose $x_0 \in A_1 \cap A_2$. If $n \geq 2$ Then $A_1 \cap A_2$ Is Path Connected. By The Lemma, Any Loop In S^n Is Homotopic To A Product Of Loops Lying In A_1 Or A_2. But $\pi_1(A_1) = 0 = \pi_1(A_2)$ And So Every Loop In S^n Is Nullhomotopic. //

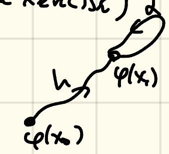
Prop: If $\varphi: X \to Y$ Is A Homotopy Equivalence, Then $\varphi_*: \pi_1(X, x_0) \to \pi_1(Y, \varphi(x_0))$ Is An Isomorphism For All $x_0 \in X$.

Pf: First Let $\varphi_t: X \to Y$ Be A Homotopy And Let h Be The Path $\varphi_t(x_0)$ In Y. I Claim $\beta_h: \pi_1(Y, \varphi_1(x_0)) \to \pi_1(Y, \varphi_0(x_0))$ Defined By $\beta_h([\alpha]) = [h \cdot \alpha \cdot h^{-1}]$ Is An Isomorphism (Exercise)

Consider The Diagram

$$\begin{tikzcd}[column sep=small] & \pi_1(Y, \varphi_1(x_0)) \arrow[d, "\beta_h"] \\ \pi_1(X, x_0) \arrow[ur, "\varphi_{1*}"] \arrow[dr, "\varphi_{0*}"'] & \pi_1(Y, \varphi_0(x_0)) \end{tikzcd}$$

I Claim This Commutes: $\varphi_{0*} = \beta_h \circ \varphi_{1*}$



To See This, Let h_t Be The Restriction Of h To $[0, t]$. Reparametrize This To $[0,1]$: $h_t(s) = h(ts)$. If α Is A Loop At x_0, Then $h_t \cdot (\varphi_t \alpha) \cdot h_t^{-1}$ Is A Homotopy Of Loops At $\varphi(x_0)$. Restricting To $t=0$ & $t=1$ Yields The Claim. Now Let $\psi: Y \to X$ Be A Homotopy Inverse Of φ: $\varphi\psi \simeq id_Y$ And $\psi\varphi \simeq id_X$. Then $\varphi_* \psi_*$ And $\psi_* \varphi_*$ Are Isomorphisms. Consider The Diagram

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "\varphi_*"] \arrow[rr, bend right, "\beta_h"'] & \pi_1(Y, \varphi(x_0)) \arrow[r, "\psi_*"] \arrow[rr, bend left, "\beta_k"] & \pi_1(X, \psi\varphi(x_0)) \arrow[r, "\varphi_*"] & \pi_1(Y, \varphi\psi\varphi(x_0)) \end{tikzcd}$$

$\Rightarrow \varphi_*$ Is An Iso. //

Free Products of Groups

Let $\{G_\alpha\}_{\alpha\in A}$ Be A Collection Of Groups. How Can We Build A Group Containing Each As A Subgroup? There Are A Couple Of Things We Might Try (eg $\prod_\alpha G_\alpha$) But The "Right" Thing To Do Is The Free Product.

Def: The Free Product $\ast_{\alpha\in A} G_\alpha$ Is The Set Consisting Of All Finite Words $g_1g_2\cdots g_m$, $m\geq 0$, Where $g_\ell \in G_{\alpha_\ell}$ And $g_\ell \neq e \in G_\alpha$, And Adjacent $g_\ell, g_{\ell+1}$ Belong To Different Groups $G_{\alpha_\ell} \neq G_{\alpha_{\ell+1}}$. This Is The Set Of Reduced Words. Define A Product By Juxtaposition:

$(g_1g_2\cdots g_m)(h_1h_2\cdots h_n) = g_1g_2\cdots g_m h_1h_2\cdots h_n$ + Then Reduce If g_m + h_1 Lie In The Same Group.

eg: $(g_1g_2)(g_2^{-1}g_1^{-1}) = g_1(g_2g_2^{-1})g_1^{-1} = g_1 e g_1^{-1} = gg_1^{-1} =$ Empty Word = Identity.

Can Check This Is Associative.

eg: $\mathbb{Z}*\mathbb{Z}$ Is An Example Of A Free Group (on 2 Generators). The Rank Is 2 In This Case.

eg: $\mathbb{Z}_2*\mathbb{Z}_2$ You Might Think This Is Finite, Or At Least All Torsion, But It Is Not. Let a, b Be The Generators Of The Two Factors: $a^2=e=b^2$. Consider The Word ab; It Has Infinite Order: $(ab)(ab)=abab\neq e$, etc. It Is Countable + You Can Write All Words:

$e, a, b, ab, ba, aba, bab, abab, baba, \ldots$ Note: $(ab)^{-1} = b^{-1}a^{-1} = ba$.

Define $\varphi: \mathbb{Z}_2*\mathbb{Z}_2 \to \mathbb{Z}_2$ By $\varphi(w) = \ell(w) \bmod 2$ ($\ell(w)$ = Length Of The Reduced Word w). This Is Clearly Surjective And $\ker\varphi = \{$Words Of Even Length$\} = \langle ab\rangle \cong \mathbb{Z}$. So $\mathbb{Z}_2*\mathbb{Z}_2$ Is The Semidirect Product Of $\underset{\langle ab\rangle}{\mathbb{Z}}$ By $\underset{\langle a\rangle}{\mathbb{Z}_2}$ $\quad a(ab)a^{-1} = ba = (ab)^{-1}$ $\quad$ Infinite Dihedral Group

Fact: $\mathbb{Z}_2 * \mathbb{Z}_3 \cong PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\langle \pm I\rangle$ $\left(SL_2(\mathbb{Z}) = \left\{\begin{bmatrix} a & b\\ c & d\end{bmatrix} \middle|\, a,b,c,d\in\mathbb{Z},\ ad-bc=1\right\}\right)$.

Universal Mapping Property

Suppose $\varphi_\alpha: G_\alpha \to H$ Is A Homomorphism. Then There Is A Unique $\bar\varphi: \ast_\alpha G_\alpha \to H$ Such That

$$\begin{tikzcd} G_\alpha \arrow[r, "i_\alpha"] \arrow[dr, "\varphi_\alpha"'] & \ast_\alpha G_\alpha \arrow[d, dotted, "\exists!\bar\varphi"] \arrow[dl, phantom, "\circlearrowleft"] \\ & H \end{tikzcd}$$

Diagram Commutes (Here $i_\alpha: G_\alpha \to \ast_\alpha G_\alpha$ Is The Obvious Inclusion)

Pf: Define $\bar\varphi(g_1\cdots g_m) = \varphi_{\alpha_1}(g_1)\cdots\varphi_{\alpha_m}(g_m)$. This Must Be The Map And Is A Homomorphism For Free. //

Free Product With Amalgamation

Let G_1 + G_2 Be Groups + Suppose $i_j: A \to G_j$ Is A Homomorphism, $j=1,2$. Consider The Diagram

$$\begin{tikzcd} A \arrow[r, "i_1"] \arrow[d, "i_2"'] & G_1 \arrow[d] \arrow[ddr, bend left, "\varphi_1"] & \\ G_2 \arrow[r] \arrow[drr, bend right, "\varphi_2"'] & G_1 *_A G_2 \arrow[dr, dotted, "\exists!\bar\varphi"] & \\ & & H \end{tikzcd}$$

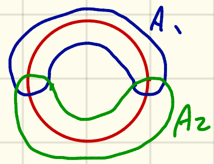
The Amalgamated Free Product $G_1 *_A G_2$ Is The Group Filling In This Diagram. To Construct It Consider $G_1 * G_2/N$ Where N = Normal Subgp Gen by $i_1(a)i_2(a)^{-1}$, $a\in A$.

e.g.: $SL_2(\mathbb{Z}) \cong \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$.

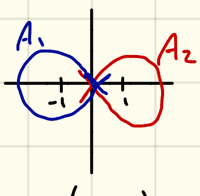
Van Kampen's Thm

Suppose X is the union of path connected open sets A_α, each containing x_0. If each $A_\alpha \cap A_\beta$ is path connected, then the map $\Phi : \ast_\alpha \pi_1(A_\alpha, x_0) \longrightarrow \pi_1(X, x_0)$ is surjective. If in addition, each $A_\alpha \cap A_\beta \cap A_\gamma$ is path connected, then $\ker \Phi$ is generated by all elements of the form $i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta, x_0)$.

Special Case $X = A_1 \cup A_2 \Rightarrow \pi_1(X) \cong \pi_1(A_1, x_0) \ast_{\pi_1(A_1 \cap A_2)} \pi_1(A_2, x_0)$

e.g.: 

$S^1 = A_1 \cup A_2$ but $A_1 \cap A_2$ not connected so can't use Thm.

e.g.: $X = \mathbb{C} \setminus \{\pm 1\}$ 

$X \simeq S^1 \vee S^1 \Rightarrow \pi_1(X) \cong \pi_1(S^1 \vee S^1)$

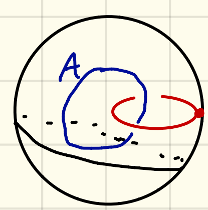
$A_1 \simeq S^1 \quad A_2 \simeq S^1 \quad A_1 \cap A_2 = \times \simeq \bullet$

$\Rightarrow \pi_1(X, 0) \cong \pi_1(A_1, 0) \ast \pi_1(A_2, 0) \cong \mathbb{Z} \ast \mathbb{Z}$.

e.g.: More generally, suppose we have spaces X_α with base points $x_\alpha \in X_\alpha$. Assume each x_α has a nbhd U_α which deformation retracts to x_α. Let $A_\alpha = X_\alpha \vee_{\beta \neq \alpha} U_\beta$. Then A_α deformation retracts to X_α + the intersection of two or more A_α is $\vee_\beta U_\beta \simeq \ast$.
So, $\Phi : \ast_\alpha \pi_1(X_\alpha) \longrightarrow \pi_1(\vee_\alpha X_\alpha)$ is an isomorphism.

Proof: We proved surjectivity when we computed $\pi_1(S^n)$. The difficult part is to compute $\ker \Phi$. This is a tedious combinatorial argument. Since we are mostly interested in the case $X = A_1 \cup A_2$ we will give a direct proof later that $\pi_1(X) \cong \pi_1(A_1) \ast_{\pi_1(A_1 \cap A_2)} \pi_1(A_2)$, at least in most cases of interest. A full proof is in Hatcher, §1.2.

e.g.: Let A be a circle in $\mathbb{R}^3$ and let $X = \mathbb{R}^3 - A$. What is $\pi_1(X)$?

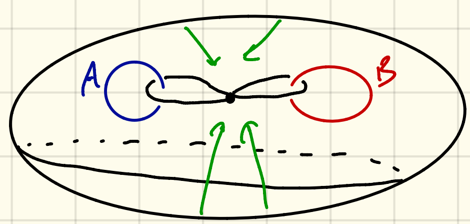


$X \simeq S^1 \vee S^2$

$$\text{So } \pi_1(X) \cong \pi_1(S^1) \ast \pi_1(S^2) \cong \mathbb{Z}$$

If B is another circle, not linked with A, then $\mathbb{R}^3 - (A \cup B) \simeq S^1 \vee S^1 \vee S^2 \vee S^2$
$\Rightarrow \pi_1(\mathbb{R}^3 - (A \cup B)) \cong \mathbb{Z} \ast \mathbb{Z}$

Read about torus knots, p.47.



Attaching Cells

Suppose X Is A Path Connected Space And Let $\varphi_\alpha : \partial e^2_\alpha \to X$ Be Attaching Maps For Some 2-Cells e^2_α. Let $Y = X \cup_\alpha e^2_\alpha$. Note That φ_α Determines A Loop In X For Each α, But The Basepoints Might Not Agree. Choose A Path γ_α From x_0 To The Basepoint Of $\varphi_\alpha(\partial e^2_\alpha)$. Then Each $\gamma_\alpha \varphi_\alpha \gamma_\alpha^{-1}$ Is A Loop At x_0. It Might Not Be Nullhomotopic, But It Will Be After e^2_α Is Attached.
Let N Be The Normal Subgroup Of $\pi_1(X,x_0)$ Generated By These $[\gamma_\alpha \varphi_\alpha \gamma_\alpha^{-1}]$. Then

$$N \leq \ker\{\pi_1(X,x_0) \to \pi_1(Y,x_0)\}$$

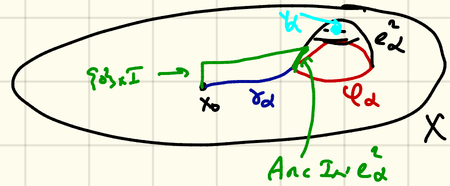
Prop: 1. The Map $\pi_1(X,x_0) \to \pi_1(Y,x_0)$ Is Surjective With Kernel N.

2. If Y Is Obtained By Attaching n-Cells To X, $n>2$, Then $\pi_1(X,x_0) \to \pi_1(Y,x_0)$ Is An Isomorphism.

3. For A Path Connected Cell Complex X, The Inclusion Of The 2-Skeleton $X^2 \hookrightarrow X$ Induces An Isomorphism $\pi_1(X^2,x_0) \xrightarrow{\cong} \pi_1(X,x_0)$.

Proof: 1. Let's Thicken Y A Bit. Attach Rectangular Strips $S_\alpha = I \times I$ To Y By Gluing $I \times \{0\}$ Along γ_α, The Edge $\{1\} \times I$ Along An Arc In e^2_α, And All The $\{0\} \times I$ Together.

Call This Space Z. Then Z Clearly Deformation Retracts To Y. In Each e^2_α Choose y_α Not On The Arc And Let $A = Z - \bigcup_\alpha \{y_\alpha\}$. Let $B = Z - X$. Then $A \simeq X$ And B Is Contractible. Since $\pi_1(B) = 0$, We



See That $\pi_1(Z) \cong \pi_1(A) / \langle \mathrm{im}\ \pi_1(A \cap B) \to \pi_1(A) \rangle$ (By Van Kampen's Thm).

What Is This Subgroup? Choose $z_0 \in A \cap B$ Near x_0 On The Segment Where All The S_α Meet And Let δ_α Be A Loop In $A \cap B$ Based At z_0 Representing $[\gamma_\alpha \varphi_\alpha \gamma_\alpha^{-1}] \in \pi_1(A, x_0)$

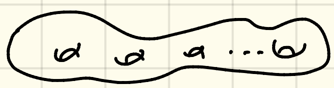
I Claim That $\pi_1(A \cap B, z_0)$ Is Generated By $\{\delta_\alpha\}$. Cover $A \cap B$ By $A_\alpha = A \cap B - \bigcup_{\beta \neq \alpha} e^2_\beta$. Then

$A_\alpha \simeq S^1$ In $e^2_\alpha - \{y_\alpha\}$ And $\pi_1(A_\alpha, z) \cong \mathbb{Z}$ Generated By δ_α And We Are Done.

2. Same As 1, Except Replace e^2_α With e^n_α. Then A_α Retracts To $S^{n-1} \Rightarrow \pi_1(A_\alpha) = 0$ For All $n>2$ & So $\pi_1(A \cap B) = 0$.

3. Follows From 2 By Induction If X Is Finite Dimensional. In General, Use The Fact That Any Loop $\gamma : I \to X$ Must Lie In Some X^n By Compactness. $2 \Rightarrow \gamma \simeq \mu : I \to X^2 \Rightarrow \pi_1(X^2)$ Surjects Onto $\pi_1(X)$. For Injectivity, If γ Is Nullhomotopic In X, Then The Image Of The Homotopy Lies In Some X^n With $n > 2$. Since $2 \Rightarrow \pi_1(X^2) \to \pi_1(X)$ Is Injective We Are Done. //

e.g.: M_g = Orientable Surface Of Genus g



The 1-Skeleton M^1_g Is Homotopic To $\bigvee_{2g} S^1$

The 2-Cell Is Attached By The Loop $\gamma = [a_1,b_1] \cdot [a_2,b_2] \cdots [a_g,b_g]$ $\quad [a_i,b_i] = a_i b_i a_i^{-1} b_i^{-1}$

Where The a_i, b_i Are The Circles In The 1-Skeleton. $\gamma : S^1 \to \bigvee_{2g} S^1$ + $\ker\{\pi_1(S^1) \to \pi_1(M_g)\}$ Is Generated By $[\gamma]$. So $\pi_1(M_g) \cong \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1,b_1] \cdots [a_g,b_g] = 1 \rangle$

Cor: If $g \neq h$, then M_g is not homotopy equivalent to M_h.

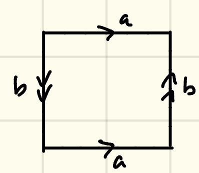
Pf: Recall the abelianization of a group G: it is the largest abelian quotient G^{ab} of G. It is constructed as the quotient $G/[G,G]$, where $[G,G]$ is generated by all $[g,h]$, $g,h \in G$. It has the obvious UMP: if $\varphi: G \to H$ is a homomorphism with H abelian, there is a unique $\bar{\varphi}: G^{ab} \to H$ making diagram commute

$$\begin{tikzcd} G \arrow[r] \arrow[dr, "\varphi"'] & G^{ab} \arrow[d, dashed, "\exists!\bar{\varphi}"] \\ & H \end{tikzcd}$$

Note that $\pi_1(M_g)^{ab} \cong \langle a_1, b_1, \dots, a_g, b_g \mid (a_1+b_1-a_1-b_1)+ \cdots + (a_g+b_g-a_g-b_g)=0 \rangle$
$\cong \mathbb{Z}^{2g}$.

So, $M_g \simeq M_h \Rightarrow \pi_1(M_g) \cong \pi_1(M_h) \Rightarrow \pi_1(M_g)^{ab} \cong \pi_1(M_h)^{ab} \Rightarrow \mathbb{Z}^{2g} \cong \mathbb{Z}^{2h} \Rightarrow 2g = 2h \Rightarrow g = h.$ //

eg: $\mathbb{RP}^2 = S^1 \cup_f e^2$, $f: \partial e^2 \to S^1$ is $z \mapsto z^2$. So $\pi_1(\mathbb{RP}^2) \cong \langle a \mid a^2 = 1 \rangle \cong \mathbb{Z}_2$.

Klein Bottle K: 

$= (S^1 \vee S^1) \cup_f e^2$, $f: S^1 \to S^1 \vee S^1$, $f \mapsto aba^{-1}b$

$\Rightarrow \pi_1(K) = \langle a, b \mid aba^{-1}b = 1 \rangle$

Cor: If G is a group, then there is a 2-dimensional cell complex X_G with $\pi_1(X_G) \cong G$.

Pf: Choose a presentation $\langle g_\alpha \mid r_\beta \rangle \cong F\langle g_\alpha \rangle / N$. Then build X_G from $\bigvee_\alpha S^1_\alpha$ by attaching 2-cells via relations r_β. //