**Covering Spaces**

**Definition:** A map \( p : E \to X \) is a **Covering Map** if each \( x \in X \) has an open neighborhood \( U \) such that \( p^{-1}(U) \) is a disjoint union \( \sqcup_k U_k \) with

1. Each \( U_k \) open in \( E \), and
2. \( p |_{U_k} : U_k \to U \) is a Homeomorphism for each \( k \).

Such an open set \( U \) is called an **Event Cover** and the \( U_k \) are called **Sheets**.

**Examples**

1. \( \text{id} : X \to X \) is a covering for any \( X \) (obviously).
2. \( E = \mathbb{R}, \quad X = S^1, \quad p(x) = e^{2\pi i x} \)
3. \( E = S^1, \quad X = S^1, \quad p(z) = z^n \)

**Note:** Any connected open \( U \subseteq S^1 \) is an Event Cover.

**Examples**

1. \( E = S^n, \quad X = \mathbb{R}P^n, \quad p : S^n \to \mathbb{R}P^n \) the **Quotient Map**. This is a Cover with \( 2^n \) Sheets (by Definition, Really).
2. \( \mathbb{R}^2 \to S^1 \times S^1 = (x, y) \to (e^{2\pi i x}, e^{2\pi i y}) \)
3. \( p : \mathbb{C} \to \mathbb{C} : z \to e^{2\pi i z} \) This is a Covering: Use Polar Coordinates \( w = p + i \theta, \quad \theta \in \mathbb{R} \) \( z \to e^{2\pi i \theta} \)
4. \( E = S^1, \quad X = \mathbb{C}^\times \to \mathbb{C}, \quad \phi(\theta) = \mathbb{C}^\times \)

**Note:** For a Covering Map \( p \), the Fibers Are Discrete.

**Unique Lifting Theorem**

Suppose \( \pi : (E, \mathcal{U}) \to (X, \mathcal{V}) \) is a covering map and \( \phi : (Y, \mathcal{W}) \to (X, \mathcal{V}) \) is continuous. If \( Y \) Is Connected, Then There Is At Most One Lift \( \phi : (Y, \mathcal{W}) \to (E, \mathcal{U}) \) \( \phi \quad \phi(\pi(\phi)) = \phi \)

**Proof:** Suppose \( \phi_1, \phi_2 \) are lifts of \( \phi \). And Set \( A = \{ x \in Y \mid \phi_1(x) = \phi_2(x) \} \). Since \( Y \) is connected, \( A \) is clopen.

**A Is Closed:** We Assume \( \text{X Hausdorff} \) (Not Necessary But It Simplifies The Argument).

The Map \( \pi : (E, \mathcal{U}) \to (X, \mathcal{V}) \) is Continuous and The Diagonal \( \Delta = \{ (x, x) \mid x \in X \} \) is Closed, Then \( \Delta = \phi_1 \circ \pi \Delta = \phi_2 \circ \pi \Delta \). Since \( \phi_1, \phi_2 \) are lifts, \( \phi_1 = \phi_2 \) on \( \Delta \).

**A Is Open:** Let \( \phi \in \text{X} \) and let \( U \) be an Event Covered Neighborhood \( \phi(y) \). \( U \) is the \( \phi^{-1}(U) = \bigcup_k U_k \). There is a Unique \( s_0 \) such that \( \phi_1(x) = \phi_2(x) = \phi(y) \). Therefore \( \phi_1(U) \cap \phi_2(U) = U_0 \). Since \( \phi \) is a map, \( \phi(U) \in \mathcal{V} \).

**Thus, A is Unique**
And since \( p \mid _{S_0} \) is injective, it follows that \( \bar{f}, \bar{f} = \bar{f}_2 \) and hence \( \bar{f}_2 \in A \). Thus \( \overline{\bar{f}} \).

Since \( Y \) is connected, we have \( A = Y \) and so \( \bar{f} \equiv \bar{f}_2 \).

**Path Lifting Theorem**

**Proof:** Uniqueness follows from the previous result. To show existence, cover the domain of \( Y \) by every covering open set. Since this set is compact, we find a partition \( 0 = t_0 < t_1 < \cdots < t_n = 1 \) of \( I \) such that \( \bar{Y} \mid ([t_{i-1}, t_i]) 
\in U_{i} \) for \( i = 1 \) to \( n \). Note that \( \bar{Y} = \bar{Y}_i = \bar{Y}_2 \cdots \bar{Y}_n \). Now \( \bar{Y} \mid V_{i-1} \rightarrow \bar{Y}_{i-1} \) is a covering. Then \( \bar{Y}_i \) lies to \( \bar{Y} \mid V_{i-1} \rightarrow \bar{Y}_i \mid V_{i-1} \rightarrow \bar{Y}_i \mid V_{i+1} \rightarrow \cdots \). Proceeding inductively, we have every \( \bar{Y}_i \) to \( \bar{Y}_n \) and then \( \bar{Y} \equiv \bar{Y}_n \) is a lift of \( \bar{Y} \).

**Covering Homotopy Theorem**

Suppose \( \overline{p} : (E, e_0) \rightarrow (X, x_0) \) is a covering map and that \( \overline{f} : Y \times I \rightarrow X \) is a homotopy.

**Proof:** 1. If all of \( X \) is eventually covered, then the result is clear.

2. For each \( y \in Y \), there is an open nbhd \( N_y \) of \( y \) and a partition \( 0 = t_0 < t_1 < \cdots < t_n = 1 \) of \( I \) such that \( p(N_y) \times ([t_{i-1}, t_i]) \) is contained in an open set \( U_{i} \) of \( X \) (by compactness of \( I \)). Let \( \bar{y}_i = \bar{f}(t, t_i) \). This forms a path in \( X \). Using the same argument as in the path lifting theorem, we have a lift \( \bar{f}_i : N_y \times [t_{i-1}, t_i] \rightarrow X \) of \( \bar{f}_i = f_i \). \( N_y \times [t_{i-1}, t_i] \), by uniqueness of lifting, \( \bar{f}_i \), and \( \bar{f}_i \) is a path on \( (N_y \times [t_{i-1}, t_i]) \). It follows that \( \bar{f}_i(t) = \bar{f}(t, t_i) \) is well-defined and continuous.

3. **Corollary:** Suppose \( \overline{f}_0, \overline{f} : (I, i_0) \rightarrow (X, x_0) \) with lifts \( \overline{f}_0, \overline{f} : (I, i_0) \rightarrow (E, e_0) \). If \( \overline{f}_0 = \overline{f}_1 \), rel \( \overline{f}_0 \), then \( \overline{f}_0 = \overline{f}_1 \), rel \( \overline{f}_0 \), respectively, \( \overline{f}_0 \) is \( \overline{f}_1 \).

4. **Proof:** By uniqueness, \( \overline{f}(t, i) = \overline{f}_0(t, i) \), i.e., \( \overline{f} = \overline{f}_0 = \overline{f}_1 \), rel \( \overline{f}_0 \).

5. **Conclusion:** The map \( \overline{p} : \overline{p}_1 : (E, e_0) \rightarrow \overline{p}_2 : (X, x_0) \) is injective.

6. **Proof:** If \( \overline{f} \) is a loop at \( e_0 \) with \( \overline{f}_0 \), then \( \overline{f} \) is a lift of \( \overline{f}_0 \). If \( \overline{f}_0 \) is a lift of \( \overline{f} \), then \( \overline{f} \equiv \overline{f}_0 \).
Q: What is the Image of $p_*$?

Note that a loop at $x_0$ lifts to a loop at $e_0$ is certainly in $\text{Image } D$. If a loop represents an element of the Image of $p_*$, then it is homotopic to a loop having such a lift. By homotopy lifting the loop itself has such a lift.

**Proof:** The number of sheets or a cover $p : (E,e_0) \to (X,x_0)$, with $E$ and $X$ path connected, equals the index of $p_* : \pi_1(E,e_0) \to \pi_1(X,x_0)$.

**Proof:** If $x$ is a loop at $x_0$, and $(x_0, H) = p_* (\pi_1(E,e_0))$, then the lift $\tilde{x}$ has the same endpoint. Since it is a loop, $\tilde{x} \equiv \tilde{x}'$ from the set of cosets of $H$ to $p^{-1}(x_0)$ by $[\tilde{x}](H) = [\tilde{x}']$. Since $E$ is path connected, $\tilde{x}$ is surjective.

Thus $\tilde{x} \equiv \tilde{x}'$ and $\tilde{x}$ lifts to a loop in $E$ based at $e_0$. $\Rightarrow \{x_0, [\tilde{x}] \} \in H$.

**Lifting Criterion.**

Suppose $p : (E,e_0) \to (X,x_0)$ is a covering space and $f : (Y,y_0) \to (X,x_0)$ is continuous with $Y$ path connected and locally path connected. Then a lift $\tilde{f} : (Y,y_0) \to (E,e_0)$ exists $\iff f_* (\pi_1(Y,y_0)) \subseteq p_* (\pi_1(E,e_0))$.

**Proof:** ($\Rightarrow$) If $\tilde{f}$ exists, then $p \circ \tilde{f} = f$ $\iff f_* (\pi_1(Y,y_0)) \subseteq p_* (\pi_1(E,e_0))$.

($\Leftarrow$) Suppose $f_* (\pi_1(Y,y_0)) \subseteq p_* (\pi_1(E,e_0))$. Let $y \in Y$ and let $f$ be a path from $y_0$ to $y$. Then $f$ has a unique lift $\tilde{f}$. Starting at $e_0$, define $\tilde{f}(y) = \tilde{f}(1)$. If $y'$ is another such path, then $f_* (\pi_1(Y,y_0)) \subseteq p_* (\pi_1(E,e_0))$. The lift $\tilde{f}'$ is a lift of $f$ starting at $e_0$ with $[\tilde{f}'] \subseteq p_* (\pi_1(E,e_0))$. Since $T$ is a homotopy $h_1$ from $h_0$ to $h$, which lifts to a loop $\tilde{h}_1$ in $E$ based at $e_0$. The homotopy lifts to $\tilde{f}'$. Since $\tilde{f}'$ is a loop at $e_0$, so is $\tilde{h}_1$. By uniqueness of lifts, $\tilde{f} = \tilde{f}'$. Via the common midpoint $\tilde{f}(1) = \tilde{f}'(1)$. So $\tilde{f}$ is well-defined.

Continuity of $f$ is not difficult. Use Local Path Connectivity.

**Note:** If $Y$ is simply connected, lifts always exist.

**Classifying Covering Spaces.**

We know $p_* : \pi_1(E,e_0) \to \pi_1(X,x_0)$ is injective.

**Q1:** Does every subgroup of $\pi_1(X,x_0)$ arise as $p_*$ for some cover? **Existence**

**Q2:** Can two different covers $X_1, X_2$ give the same subgroups? **Uniqueness**

In particular, can the trivial subgroup be realized this way? That is, does $X$ have a simply connected cover?
e8: \( X = S^1 \). \( \pi_1(S^1) = \mathbb{Z} \). The subgroups \( \langle n \rangle \) for some \( n \geq 0 \).

For \( n \geq 0 \), let \( X_n = S^1 \) with \( \mathbf{p}_n: X_n \to S^1 \). Given \( \mathbf{p}_n \), \( \mathbf{p}_n(z) = z^n \). \( \mathbf{p}_n \) is \( \pi_1 \)-injective. Then \( \mathbf{p}_n: \pi_1(X_n) \to \pi_1(X) \) is the \( \mathbb{Z} \)-module isomorphism for all \( n \geq 0 \). In particular, \( \mathbf{p}_n(\pi_1(X_n)) = \langle n \rangle \).

If \( n \neq 0 \), take \( X_0 = \mathbb{R} \). Do \( n \neq 0 \) then \( X_n \) and \( X_m \) are distinct covering spaces.

**When Can \( X \) Have a Simply Connected Cover?**

**Necessary Condition:** Each \( x \in X \) has a neighborhood \( U \) such that \( \pi_1(U, x) \to \pi_1(X, x) \) is trivial. This is called the *semi-local simple connectivity of \( X \)*.

**Why?** Suppose \( p: \tilde{X} \to X \) is a covering with \( \pi_1(\tilde{X}) = 0 \). If \( x \in X \), find an evenly covered neighborhood \( U \) of \( x \) and let \( \tilde{U} \) be a sheet. If \( y \) is a loop in \( U \), let \( \tilde{y} \) be the lift to \( \tilde{U} \). \( \tilde{y} \) is null homotopic in \( \tilde{X} \) and \( \Phi \circ p \) (null homotopy) is a null homotopy of \( \tilde{y} \) in \( \tilde{X} \).

**Claim:** \( \tilde{X} \) locally simply connected.

**Proof:** If \( X \) is path connected, locally path connected, and locally simply connected, then \( X \) has a simply connected cover \( \tilde{X} \).

**Proof:** Define \( \tilde{X} = \{ [y] \mid y \in \text{Path in } X \text{ starting at } x \} \). Here \( [y] \) denotes the \( \text{Equivalence Class} \) of \( y \). This is just a set. Define \( p: \tilde{X} \to X \) by \( p([y]) = y(x) \).

Since \( X \) is path connected, \( p \) is surjective.

**What’s the Topology on \( \tilde{X} \)?** Suppose \( [y] \in \tilde{X} \). Let \( U \) be a neighborhood \( X \) of \( y(x) \). Let \( \langle y, U \rangle = \{ [y'] \mid y \text{ is a path in } U \text{ beginning at } x(x) \} \). We may as well assume \( U \) is path connected and simply connected.

**Claim:** The sets \( \langle y, U \rangle \) form a basis for a topology on \( \tilde{X} \). For this, it suffices to show that for \( \langle y_0, U_0 \rangle \cap \langle y_1, U_1 \rangle \neq \emptyset \), there exists \( \langle y, U \rangle \) and a neighborhood \( V \) of \( y(x) \) such that \( \langle y, V \rangle \subset \langle y_0, U_0 \rangle \cap \langle y_1, U_1 \rangle \).

Then there exists \( \langle y_0, V \rangle \) in \( U_0 \) from \( y_0 \) to \( y(x) \) with \( y(x) \in V \). Also \( \langle y_1, U_1 \rangle \). Note that \( \langle y(x), U_1 \rangle \). It follows easily that \( \langle y_0, U_0 \rangle \subset \langle y(x), U_1 \rangle \).

Now \( p(\langle y, U \rangle) = \text{Path Connected Component of } U \text{ containing } y(x) \text{ and since Path Components are Open, } p \text{ is an Open M} \). Since \( p(\langle y, U \rangle) \subset U \), \( p \) is continuous.

**Claim:** \( \tilde{X} \) is a covering space. Let \( x \in X \). Since \( \tilde{X} \) is locally simply connected, \( \tilde{X} \) has a path connected, simply connected neighborhood \( U \). Then \( p^{-1}(U) \) is \( \bigcup \langle y, U \rangle \) where \( y(x) = x_0 \langle y(x), U \rangle = x \). Given two such \( \langle y_0, U_0 \rangle, \langle y_1, U_1 \rangle \) we see easily that \( \langle f(y_0), U_0 \rangle \cap \langle y_1, U_1 \rangle = \emptyset \) is \( \{ y(x) \} \). \( p^{-1}(U) \) is a disjoint union of open sets. Moreover, \( p|_{\langle y, U \rangle} \) is a Homeomorphism.
A Path from $e_0$ to $(8)$, where $Y_k: x \mapsto Y(tx)$.

**Existence**

**Proof:** Suppose $X$ is path connected, locally path connected, and semi-locally simply connected. Then for every subgroup $H \leq \pi_1(X, x_0)$, there is a covering space $p: X_H \to X$ with $p^*(\pi_1(X, x_0)) = H$ for a suitably chosen $x_0 \in X_H$.

**Proof:** Let $X$ be the simply connected cover constructed above and define a relation $\sim_H \{x\} = \{x'\}$ if $x(1) = x'(1)$ and $[x, x'] \equiv H$. This is an equivalence relation precisely because $H$ is a subgroup. Let $X_H$ be the quotient of $X$ by this relation. **Note:** If $x(1) = x'(1)$, then $[x, x'] = [H]$ for a path $\eta$. In particular, if two points in basic nbhd $\langle x, x' \rangle$ are identified, then the whole nbhd is identified. It follows that the projection $X_H \to X$ $[x] \mapsto x(1)$ is a covering. Let $p_0: X_H \to \pi_1(X_H, x_0)$ be the equivalence class of $[x_0]$. Then the image of $p_0: (X_H, x_0) \to \pi_1(X, x_0)$ is $H$. If $x$ is a loop in $X$ at $x_0$, its lift $\tilde{x}$ starting at $[x_0]$ ends at $[x']$. So the image of this lifted path in $X_H$ is a loop $\tilde{x}(\equiv [x'] \equiv [8] \in H)$.
\[\text{Def: An isomorphism of covering spaces } \pi_1: X_1 \to X, \pi_2: X_2 \to X \text{ is a homeomorphism } \]
\begin{align*}
\tilde{f}: \tilde{X}_1 \to \tilde{X}_2 \text{ with } \tilde{f}_i = \pi_i \\
\text{Proof: If } X \text{ is path connected and locally path connected, then two path connected covers } \tilde{X}_1, \tilde{X}_2 \text{ are isomorphic via } f: \tilde{X}_1 \to \tilde{X}_2 \text{ taking } \tilde{x}_i \in \pi_i^{-1}(x) \text{ to } \tilde{f}_i \in \pi_2^{-1}(x) \Leftrightarrow \pi_1 \circ (\tilde{f}_i(\tilde{x}_i, \tilde{x})) = \pi_2 \circ (\tilde{f}_i(\tilde{x}_i, \tilde{x})).
\end{align*}

\[\text{Proof (\Rightarrow): Existence of } f \Rightarrow \pi_1 = \pi_2 \circ f. \pi_2 = \pi_2 \circ f \Rightarrow \pi_1(\pi_1^{-1}(x)) = \pi_2 \circ \pi_1^{-1}(x).
\]

\[\text{Proof (\Leftarrow): Suppose the subgroups are equal. Using the lifting criterion, we lift } \tilde{f} \text{ to a map } \tilde{f}_1: (\tilde{X}_1, \tilde{x}) \to (\tilde{X}_2, \tilde{x}) \text{ with } \tilde{f}_2 \circ \tilde{f}_1 = \tilde{f}. \text{ Similarly, we get } \tilde{f}_1: (\tilde{X}_2, \tilde{x}) \to (\tilde{X}_1, \tilde{x}) \text{ with } \tilde{f}_1 \circ \tilde{f}_2 = \tilde{f}. \text{ By uniqueness of lifts, } \tilde{f}_1 \circ \tilde{f}_2 = \tilde{f} \text{ and } \tilde{f}_2 \circ \tilde{f}_1 = \tilde{f}. \text{ Since these composites fix basepoints } \tilde{x}_1, \tilde{x}_2 \text{ and } \tilde{f}_1, \tilde{f}_2 \text{ are inverse isomorphisms.}
\]

\[\text{Thm: There is a bijection between basepoint preserving isomorphism classes of path connected covering spaces } p: (\tilde{X}, \tilde{x}_0) \to (X, x_0) \text{ and the set of subgroups of } \pi_1(X, x). \text{ If basepoints are ignored, this bijection is a correspondence between iso classes of covers and conjugacy classes of subgroups of } \pi_1(X, x).
\]

\[\text{Proof: It only remains to prove the last statement. We claim that changing basepoint } \tilde{x}_0 \text{ within } \pi_1^{-1}(x) \text{ corresponds exactly to changing } \tilde{f}_0 \text{ to } \tilde{f}_0 \circ \tilde{h} \text{ for some } \tilde{h} \in \pi_1(X, \tilde{x}_0). \text{ Let } \tilde{x}_1 \text{ be another basepoint in } \pi_1^{-1}(x_0) \text{ and let } \tilde{g} \text{ be a path from } \tilde{x}_0 \text{ to } \tilde{x}_1. \text{ Then } \tilde{f}_0 \circ \tilde{g} \text{ maps a cover in } X \text{ to the corresponding cover in } X_0 \text{ with basepoint } \tilde{x}_1. \text{ It is easy to see that } \tilde{f}_0 \text{ and } \tilde{f}_0 \circ \tilde{g} \text{ are isomorphic covers of } \pi_1(X, \tilde{x}_1).
\]

\[\text{Note that if } \tilde{f}_0 \text{ is a loop at } \tilde{x}_0, \text{ then } \tilde{f}_0 \circ \tilde{g} \text{ is a loop at } \tilde{x}_1. \text{ Moreover, } \tilde{g} \text{ and } \tilde{g}^{-1} \tilde{h} \text{ in } \pi_1(X, \tilde{x}_0). \text{ Similarly, } \tilde{g} \text{ and } \tilde{g}^{-1} \tilde{h} \text{ in } \pi_1(X, \tilde{x}_1). \text{ Conversely, to change } \tilde{h} \text{ to } \tilde{h}^{-1} \tilde{g} \tilde{h} \text{ choose a loop } \tilde{g} \text{ representing } \tilde{g} \text{ and lift this to starting at } \tilde{x}_0. \text{ Any loop } \tilde{f}_0 \text{ at } \tilde{x}_0. \text{ Then } \tilde{f}_0 = \tilde{f}_0 \circ \tilde{g} \tilde{h}
\]

\[\text{Cor: A simply connected covering space is a cover of every other covering space of } X \text{. Such a space is called the universal cover (it is unique up to isomorphism).}
\]

\[\text{The action on the fiber}
\]

Let \( p: \tilde{X} \to X \) be a covering space. A path \( \gamma \) on \( X \) has a unique lift \( \tilde{\gamma} \) starting at a given point \( \tilde{x} \). Define \( \gamma: \tilde{X}(\tilde{x}) \to \tilde{X}(\tilde{x}) \) by \( \gamma(\tilde{x}) = \tilde{\gamma}(\tilde{x}) = \tilde{\gamma}(\tilde{x}) \). This is a bijection: \( \gamma \) is an isomorphism. For \( \tilde{x}, \tilde{x}' \), we have \( \gamma_{\tilde{x}, \tilde{x}'} = \gamma_{\tilde{x}', \tilde{x}} \). This reverses order of Hamiltonian class \( \tilde{x} \). We get a homomorphism \( \pi_1(X, x) \to \text{Perm}(\pi_1(X, x), \tilde{x}) \to \tilde{\gamma} \). Call this the action of \( \pi_1(X, x) \) on the fiber.
We can recover $p: \tilde{X} \to X$ from this action as follows. Let $\tilde{X} \to X$ be the
universal cover constructed earlier. Let $F = p^{-1}(x_0)$ and define $h: \tilde{X} \to X, h(\tilde{x}) = \tilde{x}(1)$, where $\tilde{x}$ is a lift of $x$ starting at $x_0$. $h$ is continuous + even. A local
homeomorphism since a neighborhood of $(\tilde{x}, \tilde{x}_0)$ in $\tilde{X} \times X$ consists of pairs $(\tilde{x} \cdot \gamma, \tilde{x}_0 \cdot \gamma)$ with $\gamma$ a path in a small neighborhood of $\gamma(1)$. $h$ is injective since $X$ is path-connected. $h$ is
almost certainly not injective. Suppose $h(\tilde{x}, \tilde{x}_0) = h(\tilde{x}', \tilde{x}_0')$. Then $\gamma$ and $\gamma'$ are paths from $\tilde{x}_0$ to the same endpoint and $\gamma_0 = \gamma_0 \gamma^{-1}(\gamma_0)$. Let $x = \gamma \gamma'$, a loop in $X$. Then $h(\tilde{x}, \tilde{x}_0) = h(x, x_0) = h(x, x_0)$. Conversely, for any loop $x$ we have $h(x, x_0) = h(x, x_0)$. Call this quotient $\tilde{X}$ and let $\tilde{X} \to \tilde{X}$ be the action. $\tilde{X}$ makes sense for any action $p$ of $\pi_1(X, x_0)$ on a set $F$: $\tilde{X} \to X, (\tilde{x}, \tilde{x}_0) \to h(\tilde{x})$

**Deck Transformations**

Let $p: \tilde{X} \to X$ be a covering. Any deck transformation of $G(X)$ the set of all isomorphisms $\tilde{X} \to X$.

This is a group under composition, called the group of deck transformations.

E.g. $p: \mathbb{R} \to S^1, G(\mathbb{R}) \cong \mathbb{Z}$ since the isomorphism is to the translations of $\mathbb{R}$ by $\mathbb{Z}$, not $\mathbb{R}$.

$p: S^1 \to S^1, z \to z^n, G(S^1) \cong \mathbb{Z}_n$ (rotations of $S^1$ through angles $2\pi/k$).

Note that by unique lifting, a deck transformation is completely determined by where

$\tilde{X}$ sends a single point. Assuming $X$ path-connected.

Def: A covering $p: \tilde{X} \to X$ is called **normal** if for each $x \in X$ and each $\tilde{x}, \tilde{x}' \in p^{-1}(x)$

there is a deck transformation taking $\tilde{x}$ to $\tilde{x}'$.

**Non-examples**

$p: \tilde{X} \to S^1 \vee S^1$ takes all nodes on

$\tilde{X}$ to the wedge point. Note

$p_1(\tilde{X}) \cong \mathbb{Z}$. Also $p_1(S^1) \to p_1(S^1)$

is $\{a\} \to \{a, b\} \subset F$.

There is no deck transformation taking $\tilde{x}$ to $\tilde{x}'$ since $\mathbb{Z}$ would be a loop at $\tilde{x}'$, but there are no non-trivial ones.
1. The covering space is normal \( \iff H \triangleleft \pi_1(X, x_0) \).

2. \( G(X) \) is isomorphic to \( N(H)/H \), where \( N(H) \) is the normalizer of \( H \) in \( \pi_1(X, x_0) \).

In particular, \( G(X) \cong \pi_1(X, x_0)/H \) if \( X \) is normal and \( H \) is the universal covering \( \tilde{X} \rightarrow X \).

Proof: Recall that a covering \( \tilde{X} \rightarrow X \) with \( \tilde{x}_0 \rightarrow x_0 \) corresponds to a covering \( \tilde{X} \rightarrow X \) of \( H \) by \( \tilde{X} \rightarrow \pi_1(X, x_0) \) where \( \tilde{X} \rightarrow \pi_1(X, x_0) \) is a group action. Then \( \tilde{X} \rightarrow \pi_1(X, x_0) \) is a homomorphism: \( \tilde{X} \rightarrow \pi_1(X, x_0) \) and \( \tilde{X} \rightarrow \pi_1(X, x_0) \) is a homomorphism that is equivalent to the existence of a deck transformation \( T \rightarrow \pi_1(X, x_0) \). So, the covering is normal \( \iff N(H) = \pi_1(X, x_0) \).

Now define \( \tilde{X} \rightarrow \pi_1(X, x_0) \) by \( \tilde{X} \rightarrow \pi_1(X, x_0) \) and \( \tilde{X} \rightarrow \pi_1(X, x_0) \) is a homomorphism: \( \tilde{X} \rightarrow \pi_1(X, x_0) \) and \( \tilde{X} \rightarrow \pi_1(X, x_0) \) is a homomorphism that is equivalent to the existence of a deck transformation \( T \rightarrow \pi_1(X, x_0) \). More generally, we have the idea of a group action. Let \( G \) be a group and \( Y \) a space. An action of \( G \) on \( Y \) is a homomorphism \( \tilde{G} \rightarrow \text{Hom}(Y) \), where \( \tilde{G} \rightarrow \text{Hom}(Y) \) for \( \tilde{g}(y) = \tilde{g}(y) \) \( \forall y, \tilde{g} \in G, y \in Y \). We usually assume \( \tilde{g} \) is injective.

Useful Condition for Actions

Every \( y \in Y \) has a neighborhood \( U \) such that all images \( g(y) \) for \( g \in G \) are distinct, i.e., \( g_1(U) \cap g_2(U) \neq \emptyset \iff g_1 = g_2 \).

**Example:** \( G \) acts on \( X \): Suppose \( U \subset X \) projects homomorphically to \( X \). Then \( g(U) \cap g(U) \neq U \), where \( g(U) = g(U) \) for some \( U, \tilde{g} \in U \). But since \( g(U) \) and \( g(U) \) lie in same \( \tilde{g}(g(U)) \) and \( \tilde{g}(g(U)) \) consists of a single point, \( g(U) \). Then \( g_1(\tilde{g}(U)) \) is a point and so \( g_1 = g_2 \).

Given an action, we can form the quotient space \( Y/G \): \( y \rightarrow g(y), g \in G \). The points of \( Y/G \) are the orbits \( Gy = \{ \tilde{g}(y) \mid g \in G \} \).

**Example:** For a normal covering \( \tilde{X} \rightarrow X \), \( \tilde{X}/G(X) \rightarrow X \).

**Example:** \( \mathbb{Z}_2 \) acts on \( S^n \): \( X \rightarrow X \), \( n/\mathbb{Z}_2 \rightarrow \mathbb{R}n \) and this action satisfies the condition since it is in the open upper hemisphere \( U \), \( g(U) \cap U = \emptyset \).
**Galois Correspondence**

Let \( G = \text{Symmetry Group of This Grid} \). \( G \) contains a copy of \( \mathbb{Z} \times \mathbb{Z} : (x, y) \mapsto (x + 1, y + 2) \). Call This Subgroup \( H \).

**Bott Lemma:** \( \gamma \) is the Glue Reflection: Translate up 1 unit + Reflect Across Vertical Line.

**Conclusion:** The Identity Takes a Square to Itself So This Action is Nice. Note The Following:

1. \( \mathbb{R}^2 / G \) is the Klein Bottle
2. \( H \) has index 2 in \( G \) so \( H \subset G \). \( \mathbb{R}^2 / H = T \) and \( \mathbb{R}^2 / H \to \mathbb{R}^2 / G \) is a \( 2:1 \) Cover.

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**Proof:** If \( G \) acting on \( Y \) is Nice, Then

1. \( \pi : Y \to Y / G \), \( y \mapsto \gamma y \) is a Normal Covering Space.
2. If \( Y \) is Path Connected, Then \( G = G(Y) \).
3. \( G \cong \pi_1(Y / G) / \pi_1(Y) \) if \( Y \) Path Connected + Locally Path Connected.

**Proof:** Let \( U \subset Y \) be an open set satisfying the condition. Then \( \pi \) identifies all the distant homeomorphic sets \( \pi_1(U / G) / \pi_1(Y) \). To A Single Open Set \( \pi(U) / G \). By Definition of The Quotient Topology, \( \pi \) restricts To A Homeomorphism From \( g(U) \) To \( \pi(U) \) for Every \( g \in G \).

Thus, \( \pi : Y \to Y / G \) is a covering. Every \( g \in G \) Acts As A Deck Transformation + The Covering Is Normal Since \( g \circ \pi \) Takes \( \pi^{-1}(U) \) To \( \pi^{-1}(U) \). \( G = G(Y) \) with Equality If \( Y \) is Path Connected Since If \( \phi \in G(Y) \) Then For Any \( y \in Y \), \( y \) and \( \phi(y) \) Are In The Same Orbit and There Is A \( g \in G \) With \( g(y) = \phi(y) \). Since Deck Transformations Are Uniquely Determined by Action On A Single Point.

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**Theorem:** \( X \) is a covering \( S^1 \to S^1 / \mathbb{Z} = \mathbb{R} / \mathbb{Z} \) and since \( \pi_1(S^1) = 0 \), we have \( \pi_1(\mathbb{R} / \mathbb{Z}) \cong \pi_1([0, 2\pi]) / \pi_1(S^1) \cong \mathbb{Z} \).

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Notes:

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- **Example:** \( \mathbb{R}^2 / \mathbb{Z} \times \mathbb{Z} \) is The Klein Bottle

Notes:
**Def:** A Graph is a 1-Dimensional CW-Complex. A Tree is a Contractible Graph.

**Pf:** Every Connected Graph $X$ contains a Maximal Tree (a Tree containing all vertices of $X$). In fact, every Tree is contained in a Maximal Tree.

**Pf:** Actually Prove The Following: Let $X_0 \subset X$ be an arbitrary Subgraph. We will construct a Subgraph $Y \subset X$ containing all vertices of $X$ such that $X_0$ is a Deformation Retract of $Y$. Taking $X = X_0$ gives the result.

First construct $X_0 < X_1 < ...$ by letting $X_i$ be obtained from $X_{i-1}$ by attaching Closures $\overline{E_i}$ of all edges $E_i \in X - X_{i-1}$ having at least one endpoint $\in X_i$.

Note That $U_i$ is open in $X$ since a Neighborhood of a Point in $X_i$ is contained in $X_i$. Also $U_i$ is closed. Since $D$ is a Union of Closed Edges and $X$ has the Weak Topology, since $X$ is connected, $U_i = X$.

Now set $Y_0 = X_0$. Assume $Y_i < X_i$ has been constructed to contain all vertices of $X_i$. Let $Y_i$ be obtained from $Y_i$ by attaching one edge connecting each vertex of $X_i - X_{i-1}$ to $Y_i$. Let $Y = UY_i$. Then $Y_i$ retracts to $Y_i$. Doing this Retraction over $\left[\frac{1}{2n}, \frac{1}{2^{i+1}}\right]$ yields a Retraction $Y \to X_0$.

**Pf:** Let $X$ be a Connected Graph and let $T$ be a Maximal Tree. Then $\pi_1(X)$ is a Free Group with basis $\{f_a\}$ corresponding to edges $E_a$ in $X - T$.

**Pf:** Fix $x \in T$. Each $E_a$ determines a Loop in $X$ by Choosing a Path $y_a$ from $x$ to one end of $E_a$, then along $E_a$, then back to $x$. Along a Path $u_a$ ($x_a$ and $u_a$ lie in $T$).

Let $F = y_a u_a$. Since $T$ is Simply Connected, $\{F\}$ determines only one Ed. The Quotient Map $X \to X/T$ is a Homotopy Equivalence. Since $T$ is a retract of $X$, the Quotient Map $X \to X/T$ is a Homotopy Equivalence. Since $T$ is a retract of $X$, the Quotient Map $X \to X/T$ is a Homotopy Equivalence.
Theorem: Every Subgroup of a Free Group is Free.

Proof: Given a free group $F$, choose a graph $X$ with $\pi_1 X = F$. If $G \leq F$ is a subgroup, there is a covering space $p : \tilde{X} \rightarrow X$ with $p_*(\pi_1(\tilde{x})) = G \cong \pi_1(X) \cong G$. Since $X$ is a graph, $\pi_1(X)$ is free, so $G$ is free.

Note: This is a purely algebraic result proven via topology!