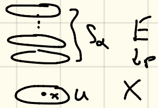
Covering Spaces

Def: A map $p: E \to X$ is a <u>covering map</u> if each $x \in X$ has an open nbhd U such that $p^{-1}(U)$ is a disjoint union $p^{-1}(U) = \coprod_\alpha S_\alpha$ with

1. Each S_α open in E, and
2. $p|_{S_\alpha}: S_\alpha \to U$ is a homeomorphism for each α.

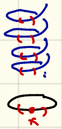
Local picture:



Such an open set U is called <u>evenly covered</u> and the S_α are called <u>sheets</u>.

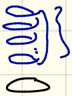
Examples 1. $id: X \to X$ is a covering for any X (obviously)

2. $E = \mathbb{R}$, $X = S^1$, $p(\alpha) = e^{2\pi i \alpha}$



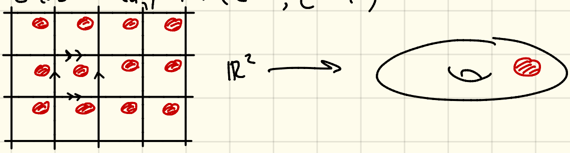
Note: Any connected open $U \neq S^1$ is evenly covered. Infinitely many sheets

3. $E = S^1$, $X = S^1$, $p: S^1 \to S^1$ $p(z) = z^n$



4. $E = S^n$, $X = \mathbb{R}P^n$, $p: S^n \to \mathbb{R}P^n$ the quotient map. This is a cover with 2 sheets (by definition, really).

5. $\mathbb{R}^2 \to S^1 \times S^1$ $(\alpha, \beta) \mapsto (e^{2\pi i \alpha}, e^{2\pi i \beta})$



6. $p: \mathbb{C} \to \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$
$w \mapsto e^{2\pi i w}$

This is a covering: Use polar coordinates $w = \rho + i\theta$, $\rho \in \mathbb{R}$, $\theta \in \mathbb{R}$
$\mathbb{C} \cong \mathbb{R}^2$, $\mathbb{C}^\times \cong \mathbb{R}^+ \times S^1$ $\rho + i\theta \mapsto (e^\rho, e^{i\theta})$

This is a covering since $\mathbb{R} \to \mathbb{R}^+$ $x \mapsto e^x$ is a homeomorphism and $\mathbb{R} \to S^1$ is a covering.

Def: The <u>fiber</u> over $y \in Y$ of a continuous map $f: X \to Y$ is $f^{-1}(y)$.

Note: For a covering map p, the fibers are discrete.

Unique Lifting Thm

Suppose $p: (E, e_0) \to (X, x_0)$ is a covering map and $f: (Y, y_0) \to (X, x_0)$ is continuous. If Y is connected, then there is <u>at most one</u> lift $\tilde{f}: (Y, y_0) \to (E, e_0)$

$$\begin{tikzcd} & (E, e_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[r, "f"] \arrow[ur, dashed, "\tilde{f}"] & (X, x_0) \end{tikzcd}$$

Proof: Suppose $\tilde{f}_1$ & $\tilde{f}_2$ are lifts of f and set $A = \{y \in Y \mid \tilde{f}_1(y) = \tilde{f}_2(y)\}$. Since $y_0 \in A$, $A \neq \emptyset$.

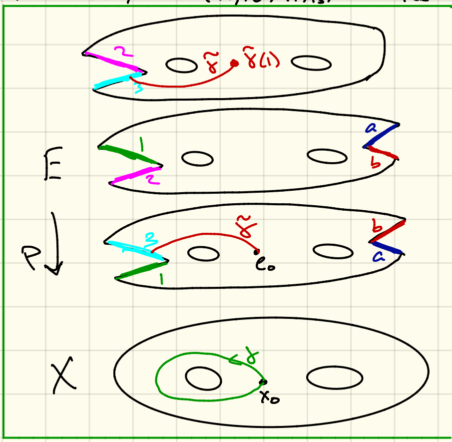
<u>A is closed:</u> We assume E Hausdorff (not necessary but it simplifies the argument). The map $\tilde{f}_1 \times \tilde{f}_2: Y \to E \times E$ is continuous and the diagonal $\Delta = \{(e,e) \mid e \in E\}$ is closed. Then $A = (\tilde{f}_1 \times \tilde{f}_2)^{-1}(\Delta)$ is closed.

<u>A is open:</u> Let $y \in A$ and let U be an evenly covered nbhd of $f(y)$. Write $p^{-1}(U) = \coprod_\alpha S_\alpha$. There is a unique α_0 such that $\tilde{f}_1(y) = \tilde{f}_2(y) \in S_{\alpha_0}$. Then $V = \tilde{f}_1^{-1}(S_{\alpha_0}) \cap \tilde{f}_2(S_{\alpha_0})$ is an open nbhd of y. If $z \in V$, then $\tilde{f}_1(z)$ and $\tilde{f}_2(z) \in S_{\alpha_0}$. Since $p \circ \tilde{f}_1(z) = f(z) = p \circ \tilde{f}_2(z)$

And since $p|_{S_{\alpha_0}}$ is injective, it follows that $\tilde{f}_1(z) = \tilde{f}_2(z)$ and hence $z \in A$. Thus $V \subseteq A$ and A is open. Since Y is connected, we have $A = Y$ and so $\tilde{f}_1 = \tilde{f}_2$. //

Path Lifting Thm

If $p: (E, e_0) \to (X, x_0)$ is a covering, then each path $\gamma: (I, 0) \to (X, x_0)$ has a unique lift $\tilde{\gamma}: (I, 0) \to (E, e_0)$.



Proof: Uniqueness follows from the previous result. To show existence, cover the image of γ by evenly covered open sets. Since this set is compact, we find a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of I such that $\gamma[t_{j-1}, t_j] \subset U_j$ and U_j is evenly covered. Set $\gamma_j = \gamma|_{[t_{j-1}, t_j]}$. Then $\gamma = \gamma_1 \cdot \gamma_2 \cdots \gamma_n$. Note that γ_1 lifts to $\tilde{\gamma}_1: [t_0, t_1] \to (E, e_0)$ since $p|_{p^{-1}(U_1)}$ is a covering. Then γ_2 lifts to $\tilde{\gamma}_2: [t_1, t_2] \to (E, \tilde{\gamma}_1(t_1))$. Proceeding inductively we lift each γ_j to $\tilde{\gamma}_j$ and then $\tilde{\gamma} = \tilde{\gamma}_1 \cdots \tilde{\gamma}_n$ is a lift of γ. //

Covering Homotopy Thm

Suppose $p: (E, e_0) \to (X, x_0)$ is a covering map and that $F: Y \times I \to X$ is a homotopy. If there is a lift $\tilde{f}_0: Y \to E$ of $f_0: Y \to X$, then there is a lift $\tilde{F}: Y \times I \to E$ such that $\tilde{F}(s, 0) = \tilde{f}_0(s)$.

Proof: 1. If all of X is evenly covered, then the result is clear.

2. For each $y \in Y$, there is an open nbhd N_y of y and a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of I such that $F(N_y \times [t_{j-1}, t_j])$ is contained in an evenly covered open set $U_j \subseteq X$ (by compactness of I). Let $\gamma_y(t) = F(y, t)$. This is a path in X. Using the same argument as in the Path Lifting Thm, we have a lift $\tilde{F}_y: N_y \times I \to E$ of $F_y = N_y \times I$. By uniqueness of lifting, $\tilde{F}_{y_1}$ and $\tilde{F}_{y_2}$ agree on $(N_{y_1} \cap N_{y_2}) \times I$. It follows that $\tilde{F}(y, t) = \tilde{F}_y(y, t)$ is well-defined and continuous. //

Cor: Suppose $\gamma_0, \gamma_1: (I, 0) \to (X, x_0)$ with lifts $\tilde{\gamma}_0, \tilde{\gamma}_1: (I, 0) \to (E, e_0)$. If $\gamma_0 \simeq \gamma_1 \text{ rel } \{0, 1\}$, then $\tilde{\gamma}_0 \simeq \tilde{\gamma}_1 \text{ rel } \{0, 1\}$. In particular, $\tilde{\gamma}_0(1) = \tilde{\gamma}_1(1)$.

Proof:

$$\begin{tikzcd} I \times \{0\} \arrow[r, "\tilde{\gamma}_0"] \arrow[d, hook] & E \arrow[d, "p"] \\ I \times I \arrow[r, "F"] \arrow[ur, "\tilde{F}"] & X \end{tikzcd}$$

By uniqueness, $\tilde{F}(s, 1) = \tilde{\gamma}_1(s)$, i.e. $\tilde{F}: \tilde{\gamma}_0 \simeq \tilde{\gamma}_1 \text{ rel } \{0, 1\}$. //

e.g. $p: S^1 \to S^1$, $z \mapsto z^2$ $\quad$ $p_*: \pi_1(S^1, 1) \to \pi_1(S^1, 1)$, $[\alpha] \mapsto 2[\alpha]$, $\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$

Cor: The map $p_*: \pi_1(E, e_0) \to \pi_1(X, x_0)$ is injective.

Pf: If $\tilde{\gamma}$ is a loop at e_0 with $p \circ \tilde{\gamma} \simeq \eta_{x_0}$, then $\tilde{\gamma}$ is a lift of $p \circ \tilde{\gamma}$ & η_{e_0} is a lift of η_{x_0} $\Rightarrow$ $\tilde{\gamma} \simeq \eta_{e_0} \text{ rel } \{0, 1\}$. //

Q: What Is The Image Of p_*?

Note That A Loop At x_0 Lifting To A Loop At e_0 Is Certainly In The Image. If A Loop Represents An Element Of The Image Of p_*, Then It Is Homotopic To A Loop Having Such A Lift + So By Homotopy Lifting The Loop Itself Has Such A Lift.

Prop: The Number Of Sheets Of A Cover $p:(\tilde{E},e_0)\to(X,x_0)$, With E And X Path Connected, Equals The Index Of $p_*(\pi_1(\tilde{E},e_0))$ In $\pi_1(X,x_0)$.

Proof: If γ Is A Loop At x_0, And If $[\alpha]\in H=p_*(\pi_1(\tilde{E},e_0))$, Then The Lift $\tilde{\alpha}\cdot\tilde{\gamma}$ Has The Same Endpoint As $\tilde{\gamma}$ Since $\tilde{\alpha}$ Is A Loop. Define Φ From The Set Of Cosets Of H To $p^{-1}(x_0)$ By $\Phi(H[\gamma])=\tilde{\gamma}(1)$. Since $\tilde{E}$ Is Path Connected, Φ Is Surjective (e_0 Can Be Joined To Any Point In $p^{-1}(x_0)$ By A Path g Projecting To A Loop γ At x_0).

But If $\Phi(H[\gamma_1])=\Phi(H[\gamma_2])$ Then $\gamma_1\cdot\gamma_2^{-1}$ Lifts To A Loop In $\tilde{E}$ Based At e_0 $\Rightarrow$ $[\gamma_1]\cdot[\gamma_2]^{-1}\in H$. //

Lifting Criterion

Suppose $p:(\tilde{E},e_0)\to(X,x_0)$ Is A Covering Space And $f:(Y,y_0)\to(X,x_0)$ Is Continuous With Y Path Connected And Locally Path Connected. Then A Lift $\tilde{f}:(Y,y_0)\to(\tilde{E},e_0)$ Exists $\Leftrightarrow$ $f_*(\pi_1(Y,y_0))\subseteq p_*(\pi_1(\tilde{E},e_0))$.

Proof: ($\Rightarrow$) If $\tilde{f}$ Exists, Then $p\circ\tilde{f}=f\Rightarrow f_*(\pi_1(Y,y_0))\subseteq p_*(\pi_1(\tilde{E},e_0))$.

($\Leftarrow$) Suppose $f_*(\pi_1(Y,y_0))\subseteq p_*(\pi_1(X,x_0))$. Let $y\in Y$ And Let γ Be A Path From y_0 To y. Then $f\gamma$ Has A Unique Lift $\widetilde{f\gamma}$ Starting At e_0. Define $\tilde{f}(y)=\widetilde{f\gamma}(1)$. If γ' Is Another Such Path Then $(f\gamma')(f\gamma)^{-1}$ Is A Loop h_0 At x_0 With $[h_0]\in f_*(\pi_1(Y,y_0))\subseteq p_*(\pi_1(\tilde{E},e_0))$. Thus There Is A Homotopy h_t From h_0 To A Loop h_1 Which Lifts To A Loop $\tilde{h}_1$ In $\tilde{E}$ Based At e_0. The Homotopy Lifts To $\tilde{h}_t$. Since $\tilde{h}_1$ Is A Loop At e_0, So Is $\tilde{h}_0$ And By Uniqueness Of Lifts, $\tilde{h}_0$ Is $\widetilde{f\gamma'}\cdot\widetilde{f\gamma}^{-1}$ Via The Common Midpoint $\widetilde{f\gamma}(1)=\widetilde{f\gamma'}(1)$. So $\tilde{f}$ Is Well-Defined. Continuity Of $\tilde{f}$ Is Not Difficult; Use Local Path Connectivity. //

Note: If Y Is Simply Connected, Lifts Always Exist.

Classifying Covering Spaces

We Know $p_*:\pi_1(\tilde{E},e_0)\to\pi_1(X,x_0)$ Is Injective.

Q1: Does Every Subgroup Of $\pi_1(X,x_0)$ Arise As $p_*(\pi_1(\tilde{E},e_0))$ For Some Cover? Existence

Q2: Can Two Different Coverings $\tilde{X}_1$, $\tilde{X}_2$ Give The Same Subgroup? Uniqueness

In Particular, Can The Trivial Subgroup Be Realized This Way? That Is, Does X Have A Simply Connected Cover?

eg: $X = S^1$. $\pi_1(S^1) = \mathbb{Z}$ The Subgroups Are $\langle n \rangle$ For Some $n \geq 0$.
For $n > 0$, Let $X_n = S^1$ With $p_n : X_n \to S^1$ Given By $p_n(z) = z^n$. Then $p_* : \pi_1(X_n) \to \pi_1(X)$ Is The Map $x \mapsto nx$ And So $p_*(\pi_1(X_n)) = \langle n \rangle$.
If $n = 0$, Take $X_0 = \mathbb{R}$. If $n \neq m$ Then X_n + X_m Are Distinct Covering Spaces.

When Can X Have A Simply Connected Cover?

Necessary Condition: Each $x \in X$ Has A Nbhd U Such That $\pi_1(U,x) \to \pi_1(X,x)$ Is Trivial. This Is Called Semilocal Simple Connectivity Of X.

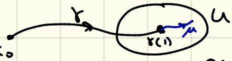
Why? Suppose $p : \tilde{X} \to X$ Is A Covering With $\pi_1(\tilde{X}) = 0$. If $x \in X$ Find An Evenly Covered Nbhd U of x And Let $\tilde{U}$ Be A Sheet. If γ Is A Loop In U, Lift It To $\tilde{\gamma} \subset \tilde{U}$; $\tilde{\gamma}$ Is Nullhomotopic In $\tilde{X}$ And Then $p \circ$ (Nullhomotopy) Is A Nullhomotopy Of γ In X.

eg: X Locally Simply Connected
X Locally Contractible (eg: Cell Complexes)

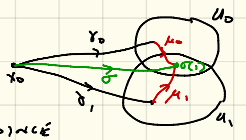
Prop: If X Is Path Connected, Locally Path Connected, And Locally Simply Connected, Then X Has A Simply Connected Cover $\tilde{X}$.

Proof: Define $\tilde{X} = \{[\gamma] \mid \gamma$ Is A Path In X Starting At $x_0\}$. Here $[\gamma]$ Denotes The Homotopy Class rel $\{0,1\}$ of γ. This Is Just A Set. Define $p : \tilde{X} \to X$ By $p([\gamma]) = \gamma(1)$. Since X Is Path Connected, p Is Surjective.

What's The Topology On $\tilde{X}$? Suppose $[\gamma] \in \tilde{X}$. Let U Be A Nbhd In X of $\gamma(1)$. Let
$\langle \gamma, U \rangle = \{[\gamma \cdot \mu] \mid \mu$ Is A Path In U Beginning At $\gamma(1)\}$. We May As Well Assume U Is Path Connected And Simply Connected.



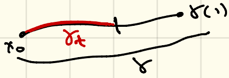
Claim: The Sets $\langle \gamma, U \rangle$ Form A Basis For A Topology On $\tilde{X}$. For This, It Suffices To Show That If $\langle \gamma_0, U_0 \rangle \cap \langle \gamma_1, U_1 \rangle \neq \emptyset$, Then There Exists $[\alpha] \in \tilde{X}$ And A Nbhd V of $\alpha(1)$ Such That $\langle \alpha, V \rangle \subseteq \langle \gamma_0, U_0 \rangle \cap \langle \gamma_1, U_1 \rangle$. Suppose $[\sigma] \in \langle \gamma_0, U_0 \rangle \cap \langle \gamma_1, U_1 \rangle$.
Then There Exists μ_0 In U_0 From $\gamma_0(1)$ To $\sigma(1)$ With $\sigma \simeq \gamma_0 \cdot \mu_0$, And μ_1 In U_1 From $\gamma_1(1)$ To $\sigma(1)$ With $\sigma \simeq \gamma_1 \cdot \mu_1$. Note That $\sigma(1) \in U_0 \cap U_1$. It Follows Easily That $\langle \sigma, U_0 \cap U_1 \rangle \subseteq \langle \gamma_0, U_0 \rangle \cap \langle \gamma_1, U_1 \rangle$.



Now, $p(\langle \gamma, U \rangle)$ = The Path Connected Component Of U Containing $\gamma(1)$ And Since Path Components Are Open, p Is An Open Map. Since $p(\langle \gamma, U \rangle) \subseteq U$, p Is Continuous.

p Is A Covering: Let $x \in X$. Since X Is Locally Simply Connected, X Has A Path Connected, Simply Connected Nbhd U. Then $p^{-1}(U) = \bigcup_{[\gamma]} \langle \gamma, U \rangle$ With $\gamma(0) = x_0$, $\gamma(1) = x$. Given Two Such $\langle \gamma_0, U \rangle$, $\langle \gamma_1, U \rangle$, We See Easily That $\langle [\gamma_0], U \rangle \cap \langle [\gamma_1], U \rangle = \emptyset \iff [\gamma_0] = [\gamma_1]$.
$\Rightarrow p^{-1}(U)$ Is A Disjoint Union Of Open Sets. Moreover, $p|_{\langle \gamma, U \rangle}$ Is A Homeomorphism.

<u>$\tilde X$ Is Path Connected</u>: Take As Basepoint $e_0 = [\eta_{x_0}]$. The Path $t \mapsto [\gamma_t]$ Is A Path From e_0 To $[\gamma]$, Where $\gamma_t : s \mapsto \gamma(ts)$.



<u>$\tilde X$ Is Simply Connected</u>: Let α Be A Loop In $\tilde X$ At e_0. Let $\gamma = p \circ \alpha$. By Unique Lifting $\tilde\gamma = \alpha$ But $\alpha(1) = [\eta_{x_0}]$ + $\tilde\gamma(1) = [\gamma] \Rightarrow \gamma = \eta_{x_0}$. This Implies That $p_*([\alpha]) = [\eta_{x_0}]$ For All α And Since p_* Is Injective, $\pi_1(\tilde X, e_0) = 0$. //

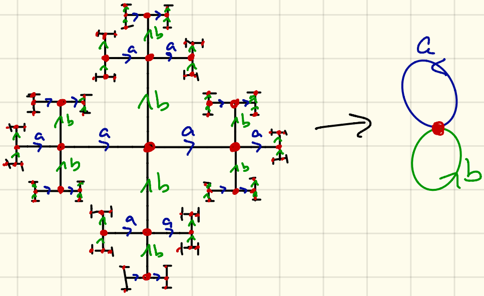
<u>Note</u>: This Is A Great Construction But It's Useless In Practice.

e.g.: $X = S^1 \vee S^1$

What Does $\tilde X$ Look Like?

Well, It Is The Homotopy Classes Of Maps Starting At Wedge Point.

The Graph Shows The First Few Iterates, But It Is Infinite.



<u>Existence</u>

<u>Prop</u>: Suppose X Is Path Connected, Locally Path Connected, And Semilocally Simply Connected. Then For Every Subgroup $H \subseteq \pi_1(X, x_0)$, There Is A Covering Space $p : X_H \to X$ With $p_*(\pi_1(X, \tilde x_0)) = H$ For A Suitably Chosen $\tilde x_0 \in X_H$.

<u>Proof</u>: Let $\tilde X$ Be The Simply Connected Cover Constructed Above And Define A Relation $\sim$ By $[\gamma] \sim [\gamma']$ If $\gamma(1) = \gamma'(1)$ And $[\gamma \cdot \gamma'^{-1}] \in H$. This Is An Equivalence Relation Precisely Because H Is A Subgroup. Let X_H Be The Quotient Of $\tilde X$ By This Relation. Note: If $\gamma(1) = \gamma'(1)$, Then $[\gamma] \sim [\gamma'] \Leftrightarrow [\gamma \cdot \eta] = [\gamma' \cdot \eta]$ For A Path η. In Particular, If Two Points In Basic Nbhds $\langle \gamma, U \rangle$ + $\langle \gamma', U' \rangle$ Are Identified, Then The Whole Nbhds Are Identified. It Follows That The Projection $X_H \to X$ $[\gamma] \mapsto \gamma(1)$ Is A Covering. Let $\tilde x_0 \in X_H$ Be The Equivalence Class Of $[\eta_{x_0}]$. Then The Image Of $p_* : \pi_1(X_H, \tilde x_0) \to \pi_1(X, x_0)$ Is H: If γ Is A Loop In X At x_0, Its Lift To $\tilde X$ Starting At $[\eta_{x_0}]$ Ends At $[\gamma]$ So The Image Of This Lifted Path In X_H Is A Loop $\Leftrightarrow$ $[\gamma] \sim [\eta_{x_0}] \Leftrightarrow [\gamma] \in H$. //

Uniqueness

Def: An Isomorphism of Covering Spaces $p_1 : \tilde{X}_1 \to X$, $p_2 : \tilde{X}_2 \to X$ Is A Homeomorphism $f : \tilde{X}_1 \to \tilde{X}_2$ with $p_1 = p_2 \circ f$.

Prop: If X Is Path Connected And Locally Path Connected, Then Two Path Connected Covers $\tilde{X}_1, \tilde{X}_2$ Are Isomorphic Via $f : \tilde{X}_1 \to \tilde{X}_2$ Taking $\tilde{x}_1 \in p_1^{-1}(x_0)$ To $\tilde{x}_2 \in p_2^{-1}(x_0)$ $\iff p_{1*}(\pi_1(\tilde{X}_1, \tilde{x}_1)) = p_{2*}(\pi_1(\tilde{X}_2, \tilde{x}_2))$.

Proof: $(\Rightarrow)$ Existence of $f \Rightarrow p_1 = p_2 \circ f$, $p_2 = p_1 \circ f^{-1} \Rightarrow p_{1*}(\pi_1(\tilde{X}_1, \tilde{x}_1)) = p_{2*}(\pi_1(\tilde{X}_2, \tilde{x}_2))$

$(\Leftarrow)$ Suppose The Subgroups Are Equal. Using The Lifting Criterion, We Lift p_1 To A Map $\tilde{p}_1 : (\tilde{X}_1, \tilde{x}_1) \to (\tilde{X}_2, \tilde{x}_2)$ With $p_2 \circ \tilde{p}_1 = p_1$. Similarly, we Get $\tilde{p}_2 : (\tilde{X}_2, \tilde{x}_2) \to (\tilde{X}_1, \tilde{x}_1)$ with $p_1 \circ \tilde{p}_2 = p_2$. By Uniqueness of Lifts, $\tilde{p}_1 \circ \tilde{p}_2 = id_{\tilde{X}_2}$ And $\tilde{p}_2 \circ \tilde{p}_1 = id_{\tilde{X}_1}$ Since These Composites Fix Basepoints. So, $\tilde{p}_1$ + $\tilde{p}_2$ Are Inverse Isomorphisms.//

Thm: There Is A Bijection Btwn Basepoint Preserving Isomorphism Classes Of Path Connected Covering Spaces $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ And The Set Of Subgroups Of $\pi_1(X, x_0)$. If Basepoints Are Ignored, This Bijection Is A Correspondence Btwn Iso Classes Of Covers And Conjugacy Classes Of Subgroups Of $\pi_1(X, x_0)$.

Proof: It Only Remains To Prove The Last Statement. We Claim That Changing Basepoint $\tilde{x}_0$ within $p^{-1}(x_0)$ Corresponds Exactly To Changing $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ To A Conjugate Subgroup. Let $\tilde{x}_1$ Be Another Basepoint In $p^{-1}(x_0)$ + Let $\tilde{\gamma}$ Be A Path From $\tilde{x}_0$ To $\tilde{x}_1$. Then $\tilde{\gamma}$ Projects To A Loop γ In X, Representing An Element $g \in \pi_1(X, x_0)$. Set $H_i = p_*(\pi_1(\tilde{X}, \tilde{x}_i))$. Note That If $\tilde{f}$ Is A Loop At $\tilde{x}_0$, $\tilde{\gamma}^{-1} \tilde{f} \tilde{\gamma}$ Is A Loop At $\tilde{x}_1$ And So $g^{-1} H_0 g \leq H_1$. Similarly, $g H_1 g^{-1} \leq H_0 \Rightarrow H_1 = g^{-1} H_0 g$. Conversely, To Change H_0 To $H_1 = g^{-1} H_0 g$, Choose A Loop γ Representing g. Lift This To $\tilde{\gamma}$ Starting At $\tilde{x}_0$ And Let $\tilde{x}_1 = \tilde{\gamma}(1)$. Then $H_1 = p_*(\pi_1(\tilde{X}, \tilde{x}_1))$.//

Cor: A Simply Connected Covering Space Is A Cover Of Every Other Covering Space Of X. Such A Space Is Called The Universal Cover (It Is Unique Up To Isomorphism).

The Action On The Fiber

Let $p : \tilde{X} \to X$ Be A Covering Space. A Path γ In X Has A Unique Lift $\tilde{\gamma}$ Starting At A Given Point In $p^{-1}(\gamma(0))$. Define $L_\gamma : p^{-1}(\gamma(0)) \to p^{-1}(\gamma(1))$ By $L_\gamma(z) = \tilde{\gamma}(1)$ with $\tilde{\gamma}(0) = z$. This Is A Bijection: $L_{\gamma^{-1}}$ Is Its Inverse. For $\gamma \cdot \eta$, We Have $L_{\gamma \cdot \eta} = L_\eta L_\gamma$. This Reversal Is Bad, So Replace L_γ By Its Inverse $p^{-1}(\gamma(1)) \to p^{-1}(\gamma(0))$. Then $L_{\gamma \cdot \eta} = L_\gamma \cdot L_\eta$. This Depends Only On Homotopy Class + So We Get A Homomorphism $\pi_1(X, x_0) \to \mathrm{Perm}(p^{-1}(x_0))$, $[\gamma] \mapsto L_\gamma$. Call This The *Action* of $\pi_1(X, x_0)$ On The *Fiber*.

We can recover $p: \tilde X \to X$ from this action as follows. Let $\tilde X_0 \to X$ be the universal cover constructed earlier. Let $F = p^{-1}(x_0)$ and define $h: \tilde X_0 \times F \to X$ by $h([\gamma], \tilde x_0) = \tilde\gamma(1)$, where $\tilde\gamma$ is a lift of γ starting at $\tilde x_0$. h is continuous + even a local homeomorphism since a nbhd of $([\gamma], \tilde x_0)$ in $\tilde X_0 \times F$ consists of pairs $([\gamma\cdot\eta], \tilde x_0)$ with η a path in a small nbhd of $\gamma(1)$. h is surjective since X is path connected. h is almost certainly not injective. Suppose $h([\gamma], \tilde x_0) = h([\gamma'], \tilde x_0')$. Then γ + γ' are paths from x_0 to the same endpoint and $\tilde x_0' = L_{\gamma'\cdot\gamma^{-1}}(\tilde x_0)$. Let $\lambda = \gamma'\cdot\gamma^{-1}$, a loop in X. Then $h([\gamma], \tilde x_0) = h([\lambda\cdot\gamma], L_\lambda(\tilde x_0))$. Conversely, for any loop λ we have $h([\gamma], \tilde x_0) = h([\lambda\cdot\gamma], L_\lambda(\tilde x_0))$.

So, h induces a map $\tilde X_0 \times F/\sim \; \to X$ where $([\gamma], \tilde x_0) \sim ([\lambda\cdot\gamma], L_\lambda(\tilde x_0))$, $[\lambda] \in \pi_1(X, x_0)$. Call this quotient $\tilde X_\rho$ where $\rho: \pi_1(X, x_0) \to \text{Perm}(F)$ is the action.

Note: $\tilde X_\rho$ makes sense for any action ρ of $\pi_1(X, x_0)$ on a set F: $\tilde X_\rho \to X$, $([\gamma], \tilde x_0) \to \gamma(1)$ is a cover.

Now, $\tilde X_\rho \to \tilde X$ is a bijection + thus a homeomorphism since h is a local homeomorphism. Since it takes fibers to fibers, it is an isomorphism. //

Deck Transformations

Let $p: \tilde X \to X$ be a covering and denote by $G(\tilde X)$ the set of all isomorphisms $\tilde X \to X$. This is a group under composition, called the group of **deck transformations**.

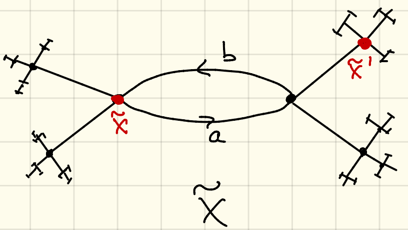
e.g. $p: \mathbb{R} \to S^1$, $G(\mathbb{R}) \cong \mathbb{Z}$ since the isomorphisms are just translations $\alpha \mapsto \alpha + n$, $n \in \mathbb{Z}$.

$p: S^1 \to S^1$, $z \mapsto z^n$, $G(\tilde X) \cong \mathbb{Z}_n$ (Rotations of S^1 through angles $2\pi k/n$)

Note that by unique lifting, a deck transformation is completely determined by where it sends a single point, assuming $\tilde X$ path connected.

Def. A covering $p: \tilde X \to X$ is called **normal** if for each $x \in X$ and each $\tilde x, \tilde x' \in p^{-1}(x)$, there is a deck transformation taking $\tilde x$ to $\tilde x'$.

Non-Example:



$p: \tilde X \to S^1 \vee S^1$ takes all nodes in $\tilde X$ to the wedge point. Note $\pi_1(\tilde X) \cong \mathbb{Z}$ and $p_*: \pi_1(\tilde X) \to \pi_1(S^1 \vee S^1)$ is $[\alpha] \mapsto [ab] \subset F_2$

There is no deck transformation g taking $\tilde x$ to $\tilde x'$ since $g_*[\alpha]$ would be a loop at $\tilde x'$, but there are no nontrivial ones.

Prop. Let $p:(\tilde{X},\tilde{x}_0)\to(X,x_0)$ Be A Nice Covering And Let $H=p_*(\pi_1(\tilde{X},\tilde{x}_0))\leq\pi_1(X,x_0)$

1. The Covering Space Is Normal $\iff H\triangleleft\pi_1(X,x_0)$.

2. $G(\tilde{X})$ Is Isomorphic To $N(H)/H$, Where $N(H)$ Is The Normalizer Of H In $\pi_1(X,x_0)$. In Particular, $G(\tilde{X})\cong\pi_1(X,x_0)/H$ If $\tilde{X}$ Is Normal And For The Universal Cover $\tilde{X}\to X$ $G(\tilde{X})\cong\pi_1(X,x_0)$.

Proof: Recall That Changing Basepoint $\tilde{x}_0\in p^{-1}(x_0)$ To $\tilde{x}_1\in p^{-1}(x_0)$ Corresponds To Conjugation Of H By $[\gamma]\in\pi_1(X,x_0)$ Where γ Lifts To A Path $\tilde{\gamma}$ From $\tilde{x}_0$ To $\tilde{x}_1$. So, $[\gamma]\in N(H)\iff$ $p_*(\pi_1(\tilde{X},\tilde{x}_0))=p_*(\pi_1(\tilde{X},\tilde{x}_1))$, Which By The Lifting Criterion Is Equivalent To The Existence Of A Deck Transformation Taking $\tilde{x}_0$ To $\tilde{x}_1$. So The Covering Is Normal $\iff N(H)=\pi_1(X,x_0)$.

Now Define $\varphi:N(H)\to G(\tilde{X})$ By $\varphi([\gamma])=\tau$, Where τ Takes $\tilde{x}_0$ To $\tilde{x}_1$. Then φ Is A Homomorphism: If $\gamma'\leftrightarrow\tau'$ Taking $\tilde{x}_0$ To $\tilde{x}_1'$ Then $\gamma\cdot\gamma'$ Lifts To $\tilde{\gamma}\cdot(\tau\cdot(\tilde{\gamma}'))$, A Path From $\tilde{x}_0$ To $\tau(\tilde{x}_1')=\tau\tau'(\tilde{x}_0)\Rightarrow\tau\tau'$ Corresponds To $[\gamma][\gamma']$. φ Is Surjective And Its Kernel Consists Of Classes $[\gamma]$ Lifting To Loops In $\tilde{X}$, i.e. $\ker\varphi=H$. //

More Generally, We Have The Idea Of A Group Action. Let G Be A Group And Y A Space. An Action Of G On Y Is A Homomorphism $\rho:G\to\mathrm{Homeo}(Y)$; Write $g:Y\to Y$ For $\rho(g)\in\mathrm{Homeo}(Y)$. Note: $g_1(g_2(y))=(g_1g_2)(y)$ $\forall g_1,g_2\in G$, $y\in Y$. We Usually Assume ρ Injective.

Useful Condition For Actions

Each $y\in Y$ Has A Nbhd U Such That All Images $g(U)$ For $g\in G$ Are Disjoint; i.e. $g_1(U)\cap g_2(U)\neq\phi\Rightarrow g_1=g_2$.

eg $G(\tilde{X})$ Acting On $\tilde{X}$: Suppose $\tilde{U}\subset\tilde{X}$ Projects Homeomorphically To X. If $g_1(\tilde{U})\cap g_2(\tilde{U})\neq\phi$, Then $g_1(\tilde{x}_1)=g_2(\tilde{x}_2)$ For Some $\tilde{x}_1,\tilde{x}_2\in\tilde{U}$. But Since $\tilde{x}_1,\tilde{x}_2$ Lie In Same $p^{-1}(x)$ And $p^{-1}(x)\cap\tilde{U}$ Consists Of A Single Point, $\tilde{x}_1=\tilde{x}_2$. Then $g_1^{-1}g_2$ Fixes This Point And So $g_1=g_2$. //

Given An Action, We Can Form The Quotient Space Y/G: $y\sim g(y)$, $g\in G$. The Points Of Y/G Are The Orbits $Gy=\{gy\mid g\in G\}$.

eg: For A Normal Covering $\tilde{X}\to X$, $\tilde{X}/G(\tilde{X})\cong X$.

eg: $\mathbb{Z}_2$ Acts On S^n: $x\mapsto -x$ $\quad S^n/\mathbb{Z}_2=\mathbb{R}P^n$ And This Action Satisfies The Condition Since If x Is In The Open Upper Hemisphere U, $g(U)\cap U=\phi$.

Prop: If G Acting On Y Is Nice, Then

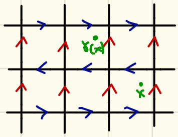
1. $p: Y \to Y/G$ $\quad y \mapsto Gy$ Is A Normal Covering Space.
2. If Y Is Path Connected, Then $G = G(Y)$.
3. $G \cong \pi_1(Y/G)/p_*(\pi_1(Y))$ If Y Path Connected + Locally Path Connected.

Proof: 1. Let $U \subset Y$ Be An Open Set Satisfying The Condition. Then p Identifies All The Disjoint Homeomorphic Sets $\{g(U) \mid g \in G\}$ To A Single Open Set $p(U) \subset Y/G$. By Definition Of The Quotient Topology, p Restricts To A Homeomorphism From $g(U)$ To $p(U)$ For Each $g \in G$. Thus, $p: Y \to Y/G$ Is A Covering. Each $g \in G$ Acts As A Deck Transformation + The Covering Is Normal Since $g_2 g_1^{-1}$ Takes $g_1(U)$ To $g_2(U)$. $G \leq G(Y)$ With Equality If Y Is Path Connected Since If $f \in G(Y)$ Then For Any $y \in Y$, y And $f(y)$ Are In The Same Orbit And There Is A $g \in G$ With $g(y) = f(y) \Rightarrow f = g$ Since Deck Transformations Are Uniquely Determined By Action On A Single Point. //

eg: $\mathbb{Z}_2$ Acting On S^n $\quad x \mapsto -x$

This Is A Covering $S^n \to S^n/\mathbb{Z}_2 = \mathbb{RP}^n$ And Since $\pi_1(S^n) = 0$, We Have $\pi_1(\mathbb{RP}^n) \cong \pi_1(\mathbb{RP}^n)/\pi_1(S^n) \cong \mathbb{Z}_2$.

eg: $\mathbb{R}^2$



Let G = Symmetry Group Of This Grid. G Contains A Copy Of $\mathbb{Z} \times \mathbb{Z}$: $(x, y) \mapsto (x+m, y+2n)$; Call This Subgroup H. But There's More: γ Is The Glide Reflection: Translate Up 1 Unit + Reflect Across Vertical Line.

Only The Identity Takes A Square To Itself So This Action Is Nice. Note The Following:

1. $\mathbb{R}^2/G$ Is The Klein Bottle
2. H Has Index 2 In G + So $H \trianglelefteq G$. $\mathbb{R}^2/H = T$ And $\underset{T}{\mathbb{R}^2/H} \to \underset{K}{\mathbb{R}^2/G}$ Is A 2:1 Cover

Galois Correspondence

Univ. Cover

$$\begin{array}{ccl} \tilde{X} & \longleftrightarrow & \pi_1(\tilde{X}) = 0 \\ \downarrow & & \\ X' & \longleftrightarrow & \pi_1(X') = H \\ \downarrow & & \\ X & \longleftrightarrow & \pi_1(X) = G \end{array}$$

$X' = \tilde{X}/H$

$G(X') \cong N(H)/H$
$= G/H$ If X' Normal

$$\left\{\begin{matrix}\text{Iso Classes Of} \\ \text{Path Ctd } X' \to X\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{Conjugacy Classes} \\ \text{Of Subgps Of } \pi_1(X)\end{matrix}\right\}$$

Fields

$$\begin{array}{c} F \\ | \\ L \\ | \\ K \end{array}$$

$Gal(F/L)$

$Gal(L/K)$

L/K Galois
$\Leftrightarrow Gal(F/L) \trianglelefteq Gal(F/K)$

$$\left\{\text{Intermediate Fields}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Subgroups Of} \\ Gal(F/K)\end{matrix}\right\}$$

$$L \longmapsto H = \{g \in Gal(F/K) \mid g(\ell) = \ell \ \ \forall \ell \in L\}$$

$$F^H \longleftarrow\!\shortmid H$$

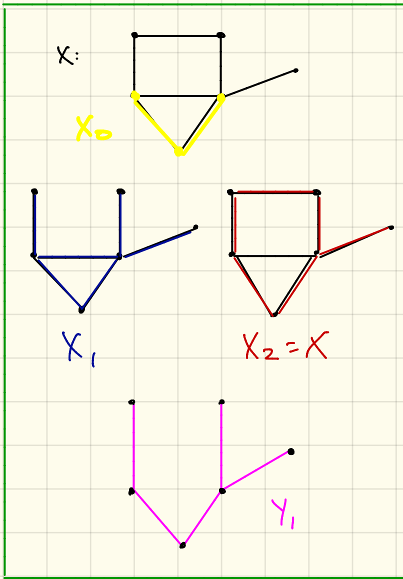
Def: A Graph Is A 1-Dimensional CW-Complex. A Tree Is A Contractible Graph.

Prop: Every Connected Graph X Contains A Maximal Tree (A Tree Containing All Vertices Of X). In Fact, Every Tree Is Contained In A Maximal Tree.

Proof: Actually Prove The Following: Let $X_0 \subset X$ Be An Arbitrary Subgraph. We Will Construct A Subgraph $Y \subset X$ Containing All Vertices Of X Such That X_0 Is A Deformation Retract Of Y. Taking $X = \{x_0\}$ Yields The Result.

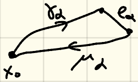
First Construct $X_0 \subset X_1 \subset \cdots$ By Letting X_{i+1} Be Obtained From X_i By Attaching Closures $\bar{e}_\alpha$ Of All Edges $e_\alpha \in X - X_i$ Having At Least One Endpoint In X_i. Note That $\bigcup X_i$ Is Open In X Since A Nbhd Of A Point In X_i Is Contained In X_{i+1}. Also $\bigcup X_i$ Is Closed Since It Is A Union Of Closed Edges And X Has The Weak Topology. Since X Is Connected, $\bigcup X_i = X$.

Now, Set $Y_0 = X_0$. Assume $Y_i \subset X_i$ Has Been Constructed To Contain All Vertices In X_i. Let Y_{i+1} Be Obtained From Y_i By Adjoining One Edge Connecting Each Vertex Of $X_{i+1} \setminus X_i$ To Y_i. Let $Y = \bigcup Y_i$. Then Y_{i+1} Retracts To Y_i. Doing This Retraction Over $[\frac{1}{2^{i+1}}, \frac{1}{2^i}]$ Yields A Retraction $Y \to X_0$.//



Prop: Let X Be A Connected Graph And Let T Be A Maximal Tree. Then $\pi_1(X)$ Is A Free Group With Basis $[f_\alpha]$ Corresponding To Edges e_α In $X - T$.

Proof: Fix $x_0 \in T$. Each e_α Determines A Loop In X By Choosing A Path γ_α From x_0 To One End Of e_α, Then Along e_α, Then Back To x_0 Along A Path μ_α (γ_α And μ_α Lie In T). Let $f_\alpha = \gamma_\alpha \cdot e_\alpha \cdot \mu_\alpha$. Since T Is Simply Connected, $[f_\alpha]$ Depends Only On e_α. The Quotient Map $X \to X/T$ Is A Homotopy Equivalence Since $T \simeq \{x_0\}$. But X/T Is A Graph With One Vertex; That Is, X/T Is A Wedge Of Circles And $\pi_1(X/T)$ Is A Free Group With Basis Given By The Images Of $[f_\alpha]$.//



Prop: Every Covering Space Of A Graph Is A Graph.

Proof: Let $p: \tilde{X} \to X$ Be A Cover. For The Vertices Of $\tilde{X}$ We Take $\tilde{X}^0 = p^{-1}(X^0)$. Note That X Is A Quotient $X = X^0 \coprod_\alpha I_\alpha$ With Each I_α Corresponding To An Edge In X. Applying Path Lifting To Each $I_\alpha \to X$ We Get A Unique Lift $I_\alpha \to \tilde{X}$ Passing Through Each Point In $p^{-1}(x)$ For $x \in I_\alpha$. These Lifts Define The Edges In $\tilde{X}$. Since $\hat{X} \to X$ Is A Local Homeomorphism The Resulting Topology On $\tilde{X}$ Is The Same As Its Original Topology.

Thm: Every Subgroup Of A Free Group Is Free.

Proof: Given A Free Group F, Choose A Graph X With $\pi_1 X \cong F$. If $G \leq F$ Is A Subgroup, There Is A Covering Space $p: \tilde{X} \to X$ With $p_*(\pi_1(\tilde{X})) = G \Rightarrow \pi_1(\tilde{X}) \cong G$. Since $\tilde{X}$ Is A Graph, $\pi_1(\tilde{X})$ Is Free $\Rightarrow G$ Is Free.

Note: This Is A Purely Algebraic Result Proven Via Topology!