Lecture 1: Quotient Spaces

Let's recall some general topology: A continuous map \( p : X \to Y \) is a quotient map if \( i \) it is surjective and \( ii \) \( U \subseteq Y \) open \( \iff p^{-1}(U) \subseteq X \) is open.

Really consider the relation ~ on \( X \) : \( x_1 x_2 \sim p(x_1) = p(x_2) \). Then \( Y \) is the set of equivalence classes. In fact, given \( X \) and a surjective function \( p : X \to Y \), we can put a topology on \( Y \) via \( U \subseteq Y \open \iff p^{-1}(U) \subseteq X \) open in \( X \) (Exercise).

Prop: (Universal Mapping Property of Quotients) Suppose \( p : X \to Y \) is a quotient map and suppose \( f : X \to \mathbb{T} \) is a continuous map with \( f(p(x)) = f(p(x_2)) \) whenever \( p(x_1) = p(x_2) \). Then there is a unique continuous \( \tilde{f} : Y \to \mathbb{T} \) with \( \tilde{f} = f \circ p \).

\[
\begin{array}{c}
X \\
\downarrow p \\
Y \\
\tilde{f}
\end{array}
\]

Proof: Define \( Y \), write \( y = p(x) \) for some \( x \subseteq X \), and then set \( \tilde{f}(y) = \tilde{f}(x) \). It is well-defined since if \( p(x_1) = y \), then \( f(p(x_1)) = f(p(x_2)) \) and hence \( \tilde{f}(y) \) is independent of the choice of \( x \). It is unique: Suppose \( g : Y \to \mathbb{T} \) satisfies \( f = g \circ p \). Then if \( y = x \), \( g(y) = g(p(x)) = f(x) = \tilde{f}(p(x)) = \tilde{f}(y) \).

Finally, \( \tilde{f} \) is continuous: Suppose \( U \subseteq Y \) is open. We must show \( \tilde{f}^{-1}(U) \) is open in \( X \). But \( \tilde{f}^{-1}(U) = f^{-1}(p^{-1}(U)) \) is open in \( X \), but \( p^{-1}(U) \) is open in \( Y \) and this is open since \( f \) is continuous.

Some Standard Spaces

- \( \mathbb{R}^n \) = \( \{ x = (x_1, \ldots, x_n) \mid x \in \mathbb{R}^n \} \)
- \( \mathbb{D}^n \) = \( \{ x \in \mathbb{R}^n \mid \|x\|_1 \leq 1 \} \) "Unit n-ball" or "n-disc"
- \( S^n \) = \( \{ x \in \mathbb{R}^{n+1} \mid \|x\|_2 = 1 \} \) (boundary of \( \mathbb{D}^{n+1} \) - ball)
- \( S^0 = \{ \pm 1 \} \subseteq \mathbb{R}^2 \)
- \( S^1 = \mathbb{S}^1 \subseteq \mathbb{C} \)
- \( S^2 = \mathbb{S}^2 \subseteq \mathbb{R}^3 \)

Examples of Quotients

1. \( \mathbb{D}^n = S^{n-1} \times \{0,1\} / S^{n-1} \times \{0\} \) Here: \( X//A \) means "Collapse A to a Point" (Really: \( a \sim b \) for all \( a \neq b \))

Strategy: (a) Put Quotient Topology on \( S^{n-1} \times \{0,1\} / S^{n-1} \times \{0\} \)
(b) Construct a Continuous Subsection \( \Psi : S^{n-1} \times \{0,1\} \to \mathbb{D}^n \) preserving ~

How: Define \( \Psi(x,t) : t \times S^{n-1} \subseteq \mathbb{D}^n \Rightarrow t \times \mathbb{D}^n \) with \( \Psi(0) = S^1 \)

Note \( \Psi(x,0) = 0 \Rightarrow \Psi(S^{n-1} \times \{0\}) = \{0\} \)

(c) \( S^{n-1} \times \{0,1\} \to S^{n-1} \times \{0,1\} / S^{n-1} \times \{0\} \)

Exercise: \( \bar{\Psi} \) is a homeomorphism.
2. \( D^n / S^{n-1} = S^n \) (Example: \( D^2 / S^1 = \mathbb{S}^2 \))

Note: A continuous surjection \( D^n \to S^n \) collapses \( \partial D^n = S^{n-1} \) to a point. Let's use

The previous example: \( D^n = S^{n-1} x \mathbb{D} \) / \( S^{n-1} x \mathbb{D} \)

\( S^{n-1} x \mathbb{D} \)

Define \( \varphi(x,t) = (\cos \pi t, \sin \pi t) x \)

Note that \( \varphi(S^{n-1} x \mathbb{D}) = (1, 0) \). So by UMP, we get \( \varphi: \frac{S^{n-1} x \mathbb{D}}{S^{n-1} x \mathbb{D}} \to S^n \) (Continuous)

Now, \( \partial D^n = S^{n-1} \) is the image of \( S^{n-1} x \mathbb{D} \) under quotient map. Note: \( \varphi(S^{n-1} x \mathbb{D}) = (1, 0) \)

By UMP again, \( \varphi \) induces \( \overline{\varphi}: \frac{D^n}{S^{n-1}} \to S^n \). It's easy to check that \( \overline{\varphi} \) is a homeo.

3. The Torus \( T = S^1 x S^1 \)

Can be obtained as an identification space as follows:

Take the unit square \([0, 1] x [0, 1]\)

Two ways to see this:

1. Products and Quotients commute:
   - If \( p_1: X_1 \to Y_1 \) and \( p_2: X_2 \to Y_2 \) are quotients, then \( p_1 x p_2: X_1 x X_2 \to Y_1 x Y_2 \) is a quotient
   - Example: \( D^1 \to S^1 \) \( D^1 \) \( x \) \( S^1 \) from Example 1 \( \Rightarrow p_1 x p_2: \mathbb{D} \) \( x \mathbb{D} \) \( \to S^1 x S^1 \) is a quotient

2. The composition of quotient maps is a quotient:

3. \( D^n \cup D^n / \partial D^n \sim \partial D^n = S^n \)
**Cell Complexes**

Let's think about the torus again:

Under this view, the holes go to two circles joined at a point:

And then the interior of the square gets attached to this. We call this a 2-cell. The two circles can be thought of as a pair of open intervals attached to a point at their ends; these are 1-cells. So we can build up \( T = S^1 \times S^1 \) in stages:

\[
\begin{array}{c}
0 \text{-cell} \quad \rightarrow \quad 2 \text{ 1-cells} \\
\end{array}
\]

We can also build the orientable surface of genus \( g \), \( M_g \), by taking a 4g-gon and identifying edges:

\[
\text{have 1 0-cell} \\
\text{2g 1-cells} \\
\text{1 2-cell}
\]

**General Construction**

A **cell complex** (or CW-complex) is a space \( X \) constructed inductively:

1. Start with a discrete set \( X^0 \), the points, and regard as 0-cells.
2. Inductively, form the \( n \)-skeleton \( X^n \) by attaching \( n \)-cells \( e^n \) via maps \( \phi_d : S^{n-1} \to D^n \) to \( X^{n-1} \). This means that \( X^n = X^{n-1} \cup D^n/\sim \)

   where \( x \sim \phi_d(x) \) for all \( x \in S^{n-1} \). \( \phi_d \) is the attaching map.
3. Stop at some \( n \), setting \( X = X^n \); or continue inductively, setting \( X = \bigcup X^n \).

In the latter case, \( X \) has the weak topology: \( A \subseteq X \) open (closed) \( \iff \) \( A \times X^n \subseteq X^n \) is open (closed) in \( X^n \) for each \( n \).

**Examples**

1. \( S^n \) has many different cell structures.

   - \( S^n = D^n / S^n \). This is \( S^n = e^n / \sim \) with \( \phi : e^n \to e^n \) the constant map.
   - Inductively build \( S^n \) from \( S^{n-1} \) by attaching two \( n \)-cells to \( S^{n-1} \), one as the northern hemisphere and one as the southern:
     
     \[
     \begin{array}{c}
     S^{n-1} \quad \phi_d : D^n \to S^{n-1} \\
     \end{array}
     \]

   Note: Have obvious inclusions \( S^0 \subseteq S^1 \subseteq \ldots \subseteq S^n \).
2. **Real Protective Space** $\mathbb{R}P^n$

As a set, $\mathbb{R}P^n = \{ \text{lines through } O \text{ in } \mathbb{R}^{n+1} \}$. It's not A All Clear how to put a reasonable topology on this set, but note: If $l$ is such a line, $l$ is determined by a nonzero vector, unique up to scalar multiplication. So:

$$\mathbb{R}P^n = \mathbb{R}^{n+1}/\sim \text{ where } v \sim \lambda v \text{ for } \lambda \neq 0.$$  

Give $\mathbb{R}P^n$ **the quotient topology.**

Another point of view: can take $v$ to be of unit length. Then $v$ and $-v$ determine the same line. Note that $v \in S^n$, so:

$$\mathbb{R}P^n = S^n/\sim \text{ where } v \sim -v \text{ (antipodal points).}$$

The topology is the same (why?). Note that this shows that $\mathbb{R}P^n$ is compact.

\[ \mathcal{O}: n=1 \mathbb{R}P^1 = \text{lines through } O \text{ in } \mathbb{R}^2 = S^1/\sim \]

A set of representatives is the upper semicircle with $1 + i$ identified:

$$-1 \rightarrow i = \mathbb{R}P^1 = S^1$$

$\mathbb{R}P^1$ is a **nonorientable surface** (like a Möbius strip). It cannot embed in $\mathbb{R}^3$.

\[ \mathcal{O}: \mathbb{R}P^n = D^n/\sim \text{ where } x \sim -x, x \in D^n = S^{n-1}. \]

Since $D^n$ with antipodal points identified is $\mathbb{R}P^{n-1}$, we see that $\mathbb{R}P^n = \mathbb{R}P^{n-1} \cup e^n$ with $\mathcal{O}: D^n = S^{n-1} \rightarrow D^n/\sim$ is the 2:1 quotient map. It follows that

$$\mathbb{R}P^n = e^0 \cup e^1 \cup \cdots \cup e^n$$

as a Cénu complex.

We also have $\mathbb{R}P^\infty = \bigcup \mathbb{R}P^n$ and so $\mathbb{R}P^\infty$ is a 2:1 quotient map.

3. **$\mathbb{C}P^n$ = Complex Protective Space.**

$\mathbb{C}P^n = \{ \text{complex lines through } O \text{ in } \mathbb{C}^{n+1} \} = \mathbb{C}^{n+1}/\sim$, $\mathbb{C}P^n \subset \mathbb{C}^{n+1}$

where $\mathbb{C}P^n = \mathbb{C}^{n+1}/\sim$, $v \sim \lambda v$ with $\lambda \in \mathbb{C}$.

$$\mathbb{C}P^n = S^{2n+1}/\sim \text{ where } v \sim \lambda v \text{ for } \lambda \neq 0.$$  

As follows:

\[ \mathcal{O}: \mathbb{C}P^n = D^{2n+1}/\sim, v \sim \lambda v, v \in D^{2n+1}. \]

Note that the vectors of $\mathbb{C}^{n+1}$ with last coordinate real and nonnegative have the form $(v, \sqrt{1 - |w|^2}) \in \mathbb{C} \times \mathbb{C}$, with $|w| \leq 1$.

$\mathbb{C}P^1 = S^2$: lines in $\mathbb{C}^2$

Described by vectors $(x,y) \in \mathbb{C}^2$ with $z^2 = 1$, where $z = x + iy$.

$\mathbb{C}P^1 = \{ (z, 1) : z \in \mathbb{C} \} = \mathbb{C} \cup \mathbb{P}^2 = S^2$. 

$\mathbb{C}P^n$ is compact.
This is the graph of the function \( \omega \rightarrow (\omega, \sqrt{1 - |\omega|^2}) \). This is a disc \( D^2 \) bounded in \( S^{2n-1} \subset S^{2n} \). Each vector in \( S^{2n} \) is equivalent to a vector in \( D^2 \) under \( v \sim u \), uniquely if the last coordinate is nonzero. If the last coordinate is 0, have identification \( v \sim u \). Then \( CP^n = \mathbb{C}P^n \cup \mathbb{C}^2n \). Also have \( CP^\infty = \mathbb{C}P^\infty \).

**Def.** A subcomplex of \( X \) is a closed subspace \( A \subset X \) that is a union of cells of \( X \). Since \( A \) is closed, each attaching map \( f_A \) for \( A \) has range in \( A \) and so \( A \) is a cell complex. A pair \((X, A)\) with \( A \) a subcomplex is called a CW-pair.

**Q1.** 1. \( K \subset X \) is a subcomplex.
2. \( RP^n \subset RP^m \) is a subcomplex.
3. \( C^n \subset C^n \).

**Homotopy**

Topology is very much concerned with understanding the homotopy type of spaces. Roughly, this means we are allowed to bend and shrink, but not glue on them.

**Q2.** \( \mathbb{R}^2 - \{0\} \) vs. \( S^1 \)

\[ f: \mathbb{R}^2 - \{0\} \to S^1 \]

This is a well-defined continuous map taking \( IR^2 - \{0\} \) onto \( S^1 \). But to think of continuously deforming \( IR^2 - \{0\} \) to \( S^1 \) we need the idea of homotopy.

**Def.** Suppose \( A \) is a subspace of \( X \). A deformation retraction of \( X \) onto \( A \) is a family of maps \( f_t: X \to X \), \( t \in [0, 1] \), such that \( f_0 = \text{id}_X \), \( f_1(x) = A \), and \( f_t|_A = \text{id}_A \) for all \( t \). The family should be continuous in the sense that the map

\[ X \times \mathbb{I} \to f_t(x) \]

is continuous.

**Q3.** \( X = IR^2 - \{0\}, \ A = S^1 \). Define \( f_t(x) = (1-t)x + \frac{x}{||x||} \). This is a deformation retraction:

\[ f_0(x) = x = f_1(x), \quad f_1(x) = \frac{x}{||x||} \quad \text{S}^1, \quad f_t(x) = (1-t)x + \frac{x}{||x||} \quad \text{CONTINUITY IS CLEAR.} \]
Def: If $f: X \to Y$ is a Homotopy Equivalence, then there is a map $g: Y \to X$ such that $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$.

Def: A Space $X$ is Contractible if $X \simeq \{\ast\}$ (one-point space).

Def: $\mathbb{R}^n$ is Contractible.

Product: Given two spaces $X$ and $Y$, the product $X \times Y$ is another space endowed with the product topology. What if $X$ and $Y$ are CW-complexes? Is $X \times Y$ a CW-complex?

Answer: Yes! The cells are products $e^n \times e^m$, $e^n$ a cell of $X$ and $e^m$ a cell of $Y$.

Example: $T = S^1 \times S^1$; $S^1 = e^0 \cup e^1$, $S^1 = e^0 \cup e^1$.

Cubes of $T$: $e^0 \times e^0 \to e^1 \times e^1$.

Usual Cell Decomposition

Quotient: Suppose $(X/A)$ is a CW-pair. The Quotient Space $X/A$ has a Cell Structure: The cells are those of $X$ minus those of $X-A$ together with one new $0$-cell, the image of $A$ in $X/A$. Attaching Maps: $e^n \to e^n$ is a cell of $X-A$ with $e^n / A \to X^n / A^n$. Then The Attaching Map for the Corresponding Cell in $X/A$ is the Composition $S_{A^n} / A \to X^n / A^n \to X^n / A^n - 1$.

Example: Let $A$ be the wedge of two circles. Then $T/A$ has a 2-cell and a 0-cell. Since there is only one way to attach a 2-cell to a 0-cell, $T/A \simeq S^2$. 
**Missing Cylinders** Let $f: X \to Y$ be a continuous map. The **mapping cylinder** is

\[ M_f = \left( Y \sqcup \{x \} \right) / \sim \quad f(x) \sim (x, 1) \]

**Picture:**

![Mapping Cylinder Diagram]

**Note:** $M_f \simeq Y$ (Exercise).

**Suspension** Let $X$ be a space. The **suspension** of $X$ is $XXI$ with $XXI \times \{0\}$ collapsing to a point and $XXI \times \{1\}$ collapsing to a point.

\[ XXI \to \quad \text{X}_{XXI} \]

$x \in X$ \implies $X \times \{0\} \approx X$ for $t \in (0, 1)$.

**Ex:** $S(S^{n-1}) = S^n \quad n \geq 1 \quad S^0 = \ast = S^{XXI} \sqcup \ast \to SX \quad \emptyset \simeq S^1$

**The Suspension Is The Union Of Two Copies Of The Cone** $CX = XXI / XXI \times \{0\}$.

If $X$ is a cell complex, so are $SX$ and $CX$. **Note:** $CX$ is contractible (Exercise).

**Join** Note that $CX$ is the union of all line segments joining points of $X$ to a point external to $X$.

Write $CX = X \times \{0\}$.

$SK = X \times S^0$.

In general, let $Y$ be another space. The collection of all line segments joining points in $X$ to points in $Y$ is called the **join**, denoted $X \star Y$.

**Formally:**

\[ X \star Y = XXI \times \{0\} \cup (X \times Y) \cup (XXI \times \{1\}) \]

\[ (x, y, 0) \sim (x, y, 0) \quad (x, y, 0) \sim (x, y, 0) \]

where $X \star Y$ are line segments, $XXI$ is a tetrahedron.

**Note** Write points of $X \star Y$ as linear combinations $t \cdot x + t \cdot y$, $0 \leq t \leq 1$, $t + t = 1$ with $0 \cdot x + 1 \cdot y, 1 \cdot x + 0 \cdot y = x$.

An **iterated join** $X_1 \star X_2 \star \ldots \star X_n$ has points that can be written as

\[ t_1 \cdot x_1 + \ldots + t_n \cdot x_n \quad 0 \leq t_i \leq 1, \quad \sum t_i = 1 \]
Even \( X_i = \{ x_i \} \)

\[ X_1 \times X_2 = \bigtriangleup \]

\[ X_1 \times X_2 \times \cdots \times X_n \]

Is a Convex Polyhedron Called the \((k-1)\)-Simplex

\[ \Delta^{n-1} = \{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid 0 \leq x_i \leq 1 \} \text{ and } \sum x_i = 1 \]. (Take \( x_i \) as \( i \)-th standard basis vectors)

\[ \text{We define } X \cup Y : \text{ Choose } x_0 \in X, y_0 \in Y, \text{ then } X \cup Y = X \cup Y / x_0 \sim y_0 \]

\[ S^1 \cup S^1 \sim \bigcirc \text{, } S^2 \cup S^2 \sim \bigoplus \]

For any cell complex \( X \), \( X^n / X^{n-1} \) is a wedge of \( n \)-simplexes \( V \cup S^n \), one for each \( n \)-cell in \( X \).

\[ \text{Smash Product } X \wedge Y = X \wedge Y / X \wedge Y \]

\[ S^n \wedge S^n = S^{n+m} : (e^m \cup e^m) \text{ has cells } (e^m \cup e^m) \text{ with cells } e^m, e^m, e^m \]

So \( S^n \wedge S^n \) has cells \( e^m \cup e^m \rightarrow S^n \wedge S^n \sim S^{n+m} \)

**Proof:** If \((X,A)\) is a CW-Pair and \( X/A \) is contractible, then \( X \rightarrow X/A \) is a Homotopy Equivalence.

**Pf:** Later.

**Graphs**

Suppose \( X = A \) a Graph with finitely many vertices \& edges.

Take a spanning Tree \( T \) (in \( \text{Re} \)). Then \( T \) is contractible (why?)

Collapse \( T \) to a Point. Then we are left with \( X/T = \bigcirc \sim S^1 \cup S^1 \)

**Attaching Spaces**

This is a generalization of the mapping cylinder. Suppose \( A \subset X \), and \( f : A \rightarrow X_0 \) is continuous. Form the quotient space

\[ X_0 \cup_\gamma X_1 = X_0 \cup X_1 / x_0 \sim f(x) \text{, } x \in A \]

We call this space \( X_0 \) with \( X_1 \) Attached Along \( A \) via \( f \). In the special case \((X_1, A) = (D^n, S^{n-1}) \) we are Attaching \( A \) as \( \downarrow \text{cell} \) to \( X_0 \) via \( f : S^{n-1} \rightarrow X_0 \).
Prove: If \((X, A)\) is a CW pair and two maps \(f, g: A \to X_0\) are homotopic, then \(X_0 U f \sim X_0 U g X_1\).

**Proof:** Consider \(S^2\) and let \(A = \{\text{two points}\} \subseteq S^2\).

Claim: \(S^2/A = S^1 U S^2\)

**Proof:** Here’s a way to get \(S^2/A\): Take an arc \(\Delta^{0,1}\) joining the two points and wrap it around a circle to identify the endpoints. Since \(X\) is contractible, this attaching map \(\Delta^{0,1}\) is homotopic to a constant map \(g\). But attaching \(S^2\) to \(S^1 U \Delta^{0,1}\) just gives \(S^1 U S^2\).

**Homotopy Extension Property**

**Q:** Suppose \(A \subset X\), \(f_0: X \to Y\) and \(f_k: A \to Y\) is a homotopy of \(f_0|A\). Can we always extend \(f_k\) to a homotopy on \(X\)?

**Def:** We say that \((X, A)\) has the Homotopy Extension Property (HEP).

**Proof:** 

\((X, A)\) has HEP \(\iff\) \(X \times I, A \times I\) is a retract of \(X \times I\).

**Proof:** 

(\(\Rightarrow\)) The HEP implies that the identity map \(X \times I, A \times I \to X \times I, A \times I\) extends to a map \(X \times I \to X \times I, A \times I\); so this subspace is a retract.

(\(\Leftarrow\)) Assume \(X\) Hausdorff (not necessary, but it simplifies matters).

First: A is closed: let \(f: X \times I \to X \times I\) be a retraction. The image \(I \subseteq \{f \subseteq X \times I\}\) is \(\{X \times I\}\).

And this is closed if \(X\) is Hausdorff. So \(X \times I, A \times I\) is closed in \(X \times I\); so \(A\) is closed in \(X\).

Now, any two maps \(X \times I, A \times I \to Y\) are homotopic on \(A \times I, A \times I\) because \(Y\) is Hausdorff. So \(f_0|A\) is a homotopy on \(A \times I\). Composing this with the retraction \(X \times I \to X \times I, A \times I\) gives an extension \(X \times I \to Y \to (X, A)\) has HEP.

**Proof:** If \((X, A)\) is a CW pair, then \(X \times I, A \times I\) is a deformation retract of \(X \times I\) and so \((X, A)\) has HEP.

**Proof:** Note that there is a retraction \(r: D^n \times I \to D^n \times I, D^n \times I\)

Set \(r_t = t r + (1-t) 1d\). This is a deformation retraction

\(r_t: D^n \times I \to D^n \times I, D^n \times I\)

This gives rise to a deformation retraction \(X^n \times I \to X^n \times I, (X^n, A^n) \times I\)

Perform this on the functions \([\frac{1}{2}, \frac{2}{3}]\) and then concatenate to get a def. retraction \(X \times I \to X \times I, A \times I\). (Were topology guarantees continuity)
**Con:** If \((X,A)\) has HE\(\rho\) and \(A \subseteq X\), then \(g : X \rightarrow X/A\) is a homotopy equiv.

**Pr:** Let \(f_t : X \rightarrow X\) be a homotopy extending a contraction of \(A\) with \(f_0 = \text{id}_X\) and \(f_t(A) \subseteq A\) for all \(t\). The composition \(g f_t : X \rightarrow X/A\) sends \(A\) to a point and so forms \(g f_t : X \rightarrow X/A\) as a composition \(X \xrightarrow{f_t} X/A\) (quotient)

where \(t = 1\), \(f_1(A) = \) a point. So \(f_1\) is a map. More \(g : X/A \rightarrow X\) with \(g f_t = f_t\) (unp)

But then we have \(g f = f\), since \(g f_t (x) = g f_t (x) = f_t(g(x)) = f_t(x)\). \(X \xrightarrow{g} X/A\)

So \(g\) and \(f\) are diverse homotopy equivalences:\n\[\begin{align*}
g = f = f_0 = \text{id}_x \\
g = f = f_0 = \text{id}_x/A \quad \Box\]