Lecture 1: Quotient Spaces

Let's recall some general topology: A continuous map $p: X \to Y$ is a **quotient map** if (i) p is surjective and (ii) $U \subseteq Y$ open $\iff p^{-1}(U) \subseteq X$ is open.

Really consider the relation $\sim$ on X: $x_1 \sim x_2 \iff p(x_1) = p(x_2)$. Then Y is the set of equivalence classes. In fact, given X and a surjective function $p: X \to Y$, we can put a topology on Y via $U \subseteq Y$ open $\iff p^{-1}(U)$ open in X (**Exercise**).

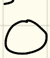
Prop: (Universal Mapping Property of Quotients) Suppose $p: X \to Y$ is a quotient map and suppose $f: X \to Z$ is a continuous map with $f(x_1) = f(x_2)$ whenever $p(x_1) = p(x_2)$. Then there is a unique continuous $\bar{f}: Y \to Z$ with $f = \bar{f} \circ p$:

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[dr, "f"'] & Y \arrow[d, dotted, "\exists!\,\bar{f}"] \\ & Z \end{tikzcd}$$

Proof: If $y \in Y$, write $y = p(x)$ for some $x \in X$ and then set $\bar{f}(y) = f(x)$. $\bar{f}$ is well-defined since if $p(x') = y$, then $f(p(x')) = f(p(x))$ and hence $\bar{f}(y)$ is independent of the choice of lift. $\bar{f}$ is unique: Suppose $g: Y \to Z$ satisfies $f = g \circ p$. Then if $y \in Y$, $g(y) = g(p(x)) = f(x) = \bar{f}(p(x)) = \bar{f}(y)$.
Finally, $\bar{f}$ is continuous: Suppose $U \subseteq Z$ is open. We must show $\bar{f}^{-1}(U)$ is open in Y. But $\bar{f}^{-1}(U)$ is open in $Y \iff p^{-1}(\bar{f}^{-1}(U))$ is open in X. But $p^{-1}(\bar{f}^{-1}(U)) = f^{-1}(U)$ and this is open since f is continuous. //

Some Standard Spaces

- $\mathbb{R}^n = \{x = (x_1, \dots, x_n) \mid x_i \in \mathbb{R}\}$, $\|x\| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$
- $D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ "Unit n-Ball" or "n-Disc"
- $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\} = \partial D^{n+1}$ (Boundary of $(n+1)$-Ball)

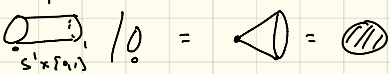
e.g. $S^0 = \{\pm 1\}$, $S^1 =$  $\subset \mathbb{R}^2$ $S^2 =$  $\subset \mathbb{R}^3$

Examples of Quotients

1. $D^n = S^{n-1} \times [0,1] / S^{n-1} \times \{0\}$

Here: X/A means "Collapse A to a point" (Really: $a \sim a'$ for all $a, a' \in A$)

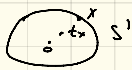
e.g. $n=2$:



Strategy (a) Put quotient topology on $S^{n-1} \times [0,1] / S^{n-1} \times \{0\}$

(b) Construct a continuous surjection $\varphi: S^{n-1} \times [0,1] \to D^n$ preserving $\sim$

How: Define $\varphi(x,t) = tx$ $\quad S^{n-1} \subset \mathbb{R}^n \Rightarrow tx \in D^n$ e.g. $n=2$

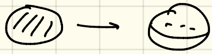


Note $\varphi(x,0) = 0 \Rightarrow \varphi(S^{n-1} \times \{0\}) = \{0\}$

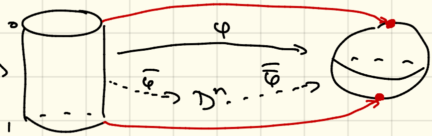
(c)

$$\begin{tikzcd} S^{n-1} \times [0,1] \arrow[r] \arrow[dr, "\varphi"'] & S^{n-1} \times [0,1] / S^{n-1} \times \{0\} \arrow[d, dotted, "\exists!\,\bar{\varphi}"] \\ & D^n \end{tikzcd}$$

Exercise: $\bar{\varphi}$ is a homeomorphism

2. $D^n / S^{n-1} \cong S^n$ e.g.: D^2/S^1:  $= S^2$

Need: A continuous surjection $D^n \to S^n$ collapsing $\partial D^n = S^{n-1}$ to a point. Let's use the previous example: $D^n = S^{n-1} \times [0,1] / S^{n-1} \times \{0\}$



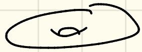
$= S^n \subset \mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{n+1}$

Define $\varphi(x,t) = (\cos(\pi t), \sin(\pi t)x)$

Note that $\varphi(S^{n-1} \times \{0\}) = (1,0)$. So by UMP, we get $\bar\varphi : \dfrac{S^{n-1}\times[0,1]}{S^{n-1}\times\{0\}} = D^n \to S^n$ (continuous)

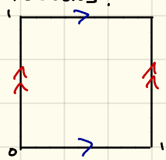
Now, $\partial D^n = S^{n-1}$ is the image of $S^{n-1} \times \{1\}$ under quotient map. Note: $\bar\varphi(S^{n-1} \times \{1\}) = (-1,0)$.

By UMP again, $\bar\varphi$ induces $\bar\varphi : D^n / S^{n-1} \to S^n$. It's easy to check that $\bar\varphi$ is a homeo.

3. The Torus $T = S^1 \times S^1$ 

Can be obtained as an identification space as follows.

Take the unit square $[0,1] \times [0,1]$

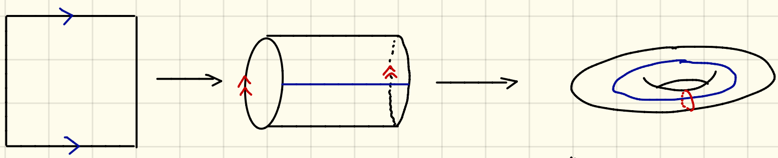


Identify $(a,0) \sim (a,1)$
$(0,b) \sim (1,b)$

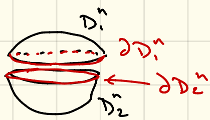
Two ways to see this:

1. Products and quotients commute:
   If $p_1 : X_1 \to Y_1$ and $p_2 : X_2 \to Y_2$ are quotients, then $p_1 \times p_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is a quotient
   e.g.: $p_j : [0,1] \to S^1$ $j = 1,2$ from Example 1 $\Rightarrow$ $p_1 \times p_2 : [0,1] \times [0,1] \to S^1 \times S^1$ is a quotient

2. The composition of quotient maps is a quotient:

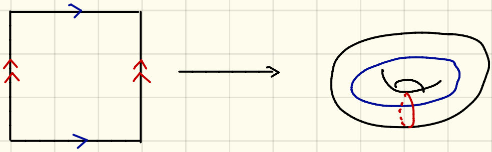


4. $D_1^n \cup D_2^n / \partial D_1^n \sim \partial D_2^n = S^n$

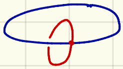


Cell Complexes

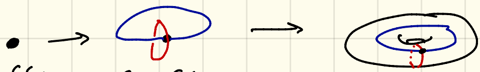
Let's Think About The Torus Again:



Under This Map, The Edges Go To Two Circles Joined At A Point:

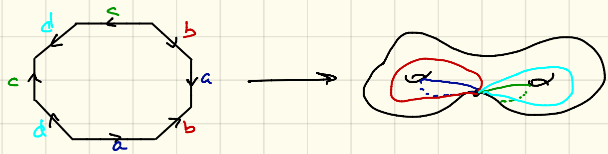


And Then The Interior Of The Square Gets Attached To This. We Call This A 2-Cell. The Two Circles Can Be Thought Of As A Pair Of Open Intervals Attached To A Point At Their Ends; These Are 1-Cells. So We Can Build Up $T = S^1 \times S^1$ In Stages:



0-Cell   2 1-Cells   1 2-Cell

We Can Also Build The Orientable Surface Of Genus g, M_g, By Taking A $4g$-Gon And Identifying Edges:



Have 1 0-Cell
$2g$ 1-Cells
1 2-Cell

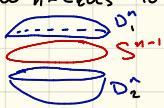
General Construction

A **Cell Complex** (or **CW-Complex**) Is A Space X Constructed Inductively:

1. Start With A Discrete Set X^0, The Points Are Regarded As 0-Cells.
2. Inductively, Form The **n-Skeleton** X^n By Attaching n-Cells e^n_α Via Maps $\varphi_\alpha : S^{n-1}_\alpha = \partial e^n_\alpha \to X^{n-1}$. This Means That $X^n = X^{n-1} \amalg_\alpha D^n_\alpha / \sim$ Where $x \sim \varphi_\alpha(x)$ For All $x \in S^{n-1}_\alpha = \partial D^n_\alpha$.
3. Stop At Some n, Setting $X = X^n$; or Continue Indefinitely, Setting $X = \bigcup_n X^n$. In The Latter Case, X Has The Weak Topology: $A \subseteq X$ open (Closed) $\iff$ $A \cap X^n$ Is Open (Closed) In X^n For Each n.

Examples

1. S^n Has Many Different Cell Structures.
   (a) $S^n = D^n / \partial D^n$. This Is $S^n = e^0 \cup e^n$ With $\varphi : \partial e^n \to e^0$ The Constant Map.
   (b) Inductively Build S^n From S^{n-1} By Attaching Two n-Cells To S^{n-1}, One As The Northern Hemisphere And One As The Southern:



$\varphi_\alpha : \partial D^n_\alpha \to S^{n-1}$ Is Identity

Note: Have Obvious Inclusions $S^0 \subset S^1 \subset S^2 \subset \dots \subset S^n$.

2. Real Projective Space $\mathbb{RP}^n$

As a set, $\mathbb{RP}^n = \{\text{Lines through } 0 \text{ in } \mathbb{R}^{n+1}\}$. It's not at all clear how to put a reasonable topology on this set, but note: if ℓ is such a line, ℓ is determined by a nonzero vector, unique up to scalar multiplication. So:

$$\mathbb{RP}^n = \mathbb{R}^{n+1} \setminus \{0\} / \sim \quad \text{where } v \sim \lambda v \text{ for } \lambda \neq 0.$$

Give $\mathbb{RP}^n$ the quotient topology.

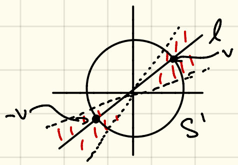
Another point of view: Can take v to be of unit length. Then v and $-v$ determine the same line. Note that $v \in S^n$, so:

$$\mathbb{RP}^n = S^n / \sim \quad \text{where } v \sim -v \text{ (Antipodal Points)}$$

The topology is the same (why?). Note that this shows that $\mathbb{RP}^n$ is Compact.

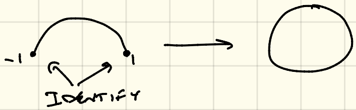
eg: $n=1$ $\quad \mathbb{RP}^1 = \text{Lines through } 0 \text{ in } \mathbb{R}^2$

$= S^1/\sim$



Open Set Around ℓ

A set of representatives is the upper semicircle with 1 & -1 identified:



$= \mathbb{RP}^1 = S^1$

$\mathbb{RP}^2$ is a nonorientable surface (like a Möbius strip). It cannot embed in $\mathbb{R}^3$.

OR: $\mathbb{RP}^n = D^n/\sim$ where $x \sim -x$, $x \in \partial D^n = S^{n-1}$. Since ∂D^n with antipodal points identified is $\mathbb{RP}^{n-1}$, we see that $\mathbb{RP}^n = \mathbb{RP}^{n-1} \cup e^n$ with $\varphi: \partial D^n = S^{n-1} \to \mathbb{RP}^{n-1}$ is the $2:1$ quotient map. It follows that

$$\mathbb{RP}^n = e^0 \cup e^1 \cup \cdots \cup e^n \quad \text{As a Cell Complex.}$$

We also have $\mathbb{RP}^\infty = \bigcup_n \mathbb{RP}^n = e^0 \cup e^1 \cup \cdots \cup e^n \cup \cdots = \text{Lines through } 0 \text{ in } \mathbb{R}^\infty = \bigcup_n \mathbb{R}^n$

And $S^\infty \to \mathbb{RP}^\infty$ is a $2:1$ quotient map.

$\mathbb{CP}^1 = S^2$: Lines in $\mathbb{C}^2$ described by vectors $\{(1,\lambda), \lambda \in \mathbb{C}\} \cup \{(0,1)\}$ (Point at ∞)

$= \mathbb{C} \cup \{\infty\} = S^2$

3. $\mathbb{CP}^n$ = Complex Projective Space.

$$\begin{aligned}
\mathbb{CP}^n &= \{\text{Complex Lines through } 0 \text{ in } \mathbb{C}^{n+1}\} \\
&= \mathbb{C}^{n+1} \setminus \{0\}/\sim, \quad v \sim \lambda v \quad \lambda \neq 0 \text{ in } \mathbb{C} \\
&= S^{2n+1}/\sim, \quad v \sim \lambda v \quad |\lambda| = 1 \implies \mathbb{CP}^n \text{ Compact}
\end{aligned}$$

OR: $\mathbb{CP}^n = D^{2n}/\sim$, $v \sim \lambda v$, $v \in \partial D^{2n}$ as follows:

Note that the vectors in $S^{2n+1} \subset \mathbb{C}^{n+1}$ with last coordinate real + nonnegative have the form $(w, \sqrt{1-|w|^2}) \in \mathbb{C}^n \times \mathbb{C}$, with $|w| \leq 1$.

This Is The Graph Of The Function $w \mapsto (w, \sqrt{1-|w|^2})$. This Is A Disc D^{2n}_+ Bounded By $S^{2n-1} \subset S^{2n+1}$ Consisting Of Vectors Of Form $(w, 0) \in \mathbb{C}^n \times \mathbb{C}$, $|w|=1$. Each Vector In S^{2n+1} Is Equivalent To A Vector In D^{2n}_+ Under $v \sim \lambda v$, Uniquely If The Last Coordinate Is Nonzero. If The Last Coordinate Is 0, Have Identifications $v \sim \lambda v$, $v \in S^{2n-1}$. Thus $\mathbb{C}P^n = \mathbb{C}P^{n-1} \cup e^{2n}$ $\quad \varphi : S^{2n-1} \to \mathbb{C}P^n$ Is Quotient.

Thus $\mathbb{C}P^n = e^0 \cup e^2 \cup \cdots \cup e^{2n}$. Also Have $\mathbb{C}P^\infty = \bigcup_n \mathbb{C}P^n$.

Def: A Subcomplex Of X Is A Closed Subspace $A \subset X$ That Is A Union Of Cells Of X. Since A Is Closed, Each Attaching Map φ_α For $e_\alpha \subset A$ Has Image In A And So A Is A Cell Complex. A Pair (X, A) With A A Subcomplex Is Called A **CW-Pair**.

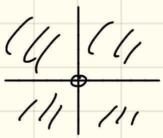
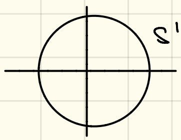
eg: 1. $X^n \subset X$ Is A Subcomplex

2. $\mathbb{R}P^m \subset \mathbb{R}P^n$ Is A Subcomplex, $m \le n$

3. $\mathbb{C}P^m \subset \mathbb{C}P^n$

Homotopy

Topology Is Very Much Concerned With Understanding The Homotopy Type Of Spaces. Roughly, This Means We Are Allowed To Bend + Shrink, But Not Glue Or Tear.

eg:  $\mathbb{R}^2 - \{0\}$ vs. 

$$f : \mathbb{R}^2 - \{0\} \to S^1$$
$$x \mapsto \frac{x}{\|x\|}$$

This Is A Well-Defined Continuous Map Taking $\mathbb{R}^2 - \{0\}$ Onto S^1! But To Think Of Continuously Deforming $\mathbb{R}^2 - \{0\}$ To S^1 We Need The Idea Of Homotopy.

Def: Suppose A Is A Subspace Of X. A **Deformation Retraction** Of X onto A Is A Family Of Maps $f_t : X \to X$, $t \in [0,1]$, Such That $f_0 = \mathrm{Id}_X$, $f_1(X) = A$, And $f_t|_A = \mathrm{Id}_A$ For All t. The Family Should Be Continuous In The Sense That The Map

$$X \times I \to X, \quad (x, t) \mapsto f_t(x)$$

Is Continuous.

eg: $X = \mathbb{R}^2 - \{0\}$, $A = S^1$. Define $f_t(x) = (1-t)x + t\frac{x}{\|x\|}$. This Is A Deformation Retraction: $f_0(x) = x = \mathrm{Id}_X(x)$; $f_1(x) = \frac{x}{\|x\|} \in S^1$, If $x \in S^1$, $\|x\| = 1 \Rightarrow f_t(x) = (1-t)x + tx = x$. Continuity Is Clear.

Def: A **Homotopy** is a map $F: X\times I \to Y$. We often write $f_t(x) = F(x,t)$. And say the maps f_0 and f_1 are **homotopic**, written $f_0 \simeq f_1$.

eg: A deformation retraction of X onto A is a homotopy from Id_X to $r: X \to A$.

More generally, a homotopy $F: X\times I \to Y$ is called a <u>homotopy relative to A</u> if $F(a,t) = F(a,0)$ for all $a \in A$, $t \in [0,1]$.

Def: A map $f: X \to Y$ is a **homotopy equivalence** if there is a map $g: Y \to X$ such that $g\circ f \simeq \mathrm{Id}_X$ and $f \circ g \simeq \mathrm{Id}_Y$. We write $X \simeq Y$ and say X + Y are **homotopy equivalent**.

Unpack this: $g\circ f: X \to X \Rightarrow \exists\, F: X\times I \to X$, $F(x,0) = g\circ f(x)$, $F(x,1) = \mathrm{Id}_X(x)$

$f\circ g: Y \to Y \Rightarrow \exists\, G: Y \times I \to Y$, $G(y,0) = f\circ g(y)$, $G(y,1) = \mathrm{Id}_Y(y)$

eg: A deformation retraction is a homotopy equivalence. So, $\mathbb{R}^2 - \{0\} \simeq S^1$.

Def: A space X is **contractible** if $X \simeq *$ (one-point space).

eg: $\mathbb{R}^n$ is contractible

Define $r: \mathbb{R}^n \to \{0\}$ by $r(x) = 0$. Define $F: \mathbb{R}^n \times I \to \mathbb{R}^n$ by $F(x,t) = (1-t)x$. Then $F(x,0) = x$ and $F(x,1) = 0 = r(x)$. So F is a homotopy from $\mathrm{id}_{\mathbb{R}^n}$ to r.

Some Operations on Cell Complexes

<u>Product</u> Given two spaces X and Y the product $X \times Y$ is another space endowed with the product topology. What if X and Y are CW-complexes? Is $X\times Y$ a CW-complex?

Answer: Yes! The cells are products $e^m_\alpha \times e^n_\beta$, e^m_α a cell in X + e^n_β a cell in Y.

eg: $T = S^1 \times S^1$: $S^1 = e^0 \cup e^1$, $S^1 = e^0 \cup e^1$

Cells of T: $e^0 \times e^0 \leftrightarrow \bullet$

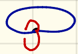
$e^0 \times e^1 \leftrightarrow$

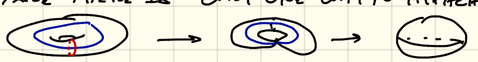
$e^1 \times e^0 \leftrightarrow$

$e^1 \times e^1 \cong e^2$

Usual cell decomposition

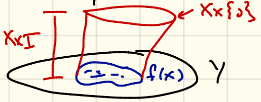
<u>Quotient</u> Suppose (X,A) is a CW-pair. The quotient space X/A has a cell structure: The cells are those of $X - A$ together with one new 0-cell, the image of A in X/A. Attaching maps: If e^n_α is a cell of $X - A$ with $\varphi_\alpha : S^{n-1}_\alpha \to X^{n-1}$, then the attaching map for the corresponding cell in X/A is the composition $S^{n-1}_\alpha \to X^{n-1} \to X^{n-1}/A^{n-1}$.

eg: Let $A \subset T$ be the wedge of two circles  Then T/A has a 2-cell and a 0-cell + since there is only one way to attach a 2-cell to a 0-cell, $T/A \simeq S^2$.



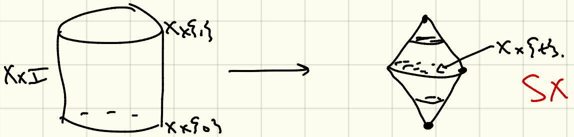
Mapping Cylinder Let $f: X \to Y$ be a continuous map. The **mapping cylinder** is $M_f = \{Y \amalg (X\times I)\}/\sim$ $\quad f(x)\sim(x,1)$

Picture:

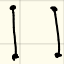


Note: $M_f \simeq Y$ (Exercise).

Suspension Let X be a space. The **suspension** of X is $X\times I$ with $X\times\{0\}$ collapsed to a point and $X\times\{1\}$ collapsed to a point.

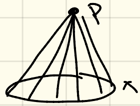


$X\times\{t\} \approx X$ for $t\in(0,1)$.

e.g.: $S(S^{n-1}) = S^n$ $\quad$ $n=1$: $S^0 = \bullet \quad \bullet$ $\quad S^0\times I$  $\to SX$  $\simeq S^1$

The suspension is the union of two copies of the **cone** $CX = X\times I/X\times\{0\}$.
If X is a cell complex, so are SX and CX. Note: CX is contractible (Exercise).

Join Note that CX is the union of all line segments joining points of X to a point external to X.



Write $CX = X * \{p\}$

$SX = X * S^0$.

In general, if Y is another space the collection of all line segments joining points in X to points in Y is called the **join**, denoted $X*Y$.

Formally, $X*Y = X\times Y\times I/\sim$ $\quad (x,y_1,0)\sim(x,y_2,0)$ $\quad$ i.e. collapse $X\times Y\times\{0\}$ to X
$(x_1,y,0)\sim(x_2,y,0)$ $\quad$ $X\times Y\times\{1\}$ to Y

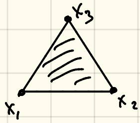
e.g. If X + Y are line segments, $X*Y$ is a tetrahedron.

Note Write points of $X*Y$ as linear combinations t_1x+t_2y, $0\le t_i\le 1$, $t_1+t_2=1$ with $0x+1y=y$, $1x+0y=x$.

An **iterated join** $X_1 * X_2 * \cdots * X_n$ has points that can be written as
$$t_1x_1+t_2x_2+\cdots+t_nx_n \qquad 0\le t_i\le 1,\ \sum_{i=1}^{n} t_i = 1.$$

eg: Each $X_i = \{x_i\}$

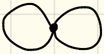
$X_1 * X_2 =$ 

$X_1 * X_2 * X_3$ 

$X_1 * X_2 * \cdots * X_n$ Is A Convex Polyhedron Called The $(n-1)$-Simplex

$\Delta^{n-1} = \{(t_1, \dots, t_n) \in \mathbb{R}^n \mid 0 \le t_i \le 1 \text{ And } \sum_{i=1}^{n} t_i = 1\}$. (Take x_i = ith Standard Basis Vector)

Wedge Sum $X \vee Y$: Choose $x_0 \in X$, $y_0 \in Y$, Then $X \vee Y = X \amalg Y / x_0 \sim y_0$

$S^1 \vee S^1$  , $S^1 \vee S^2$ 

For Any Cell Complex X, X^n / X^{n-1} Is A Wedge Of n-Spheres $\bigvee_\alpha S^n_\alpha$, One For Each n-Cell In X.

Smash Product $X \wedge Y = X \times Y / X \vee Y$

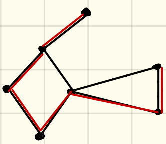
eg: $S^n \wedge S^m \approx S^{n+m}$: $(e^0 \cup e^n) \times (e^0 \cup e^m)$ Has Cells $e^0 \times e^0, e^n \times e^0, e^0 \times e^m$, $e^n \times e^m$ $\quad$ $(e^0 \cup e^n) \vee (e^0 \cup e^m)$ Has Cells e^0, e^n, e^m

So $S^n \wedge S^m$ Has Cells e^0 And $e^n \times e^m \Rightarrow S^n \wedge S^m \approx S^{n+m}$

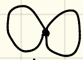
Prop: If (X, A) Is A CW-Pair And If A Is Contractible, Then $X \to X/A$ Is A Homotopy Equivalence.

Pf: Later.

eg: **Graphs** Suppose X Is A Connected Graph With Finitely Many Vertices + Edges.



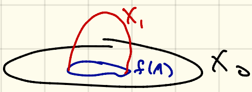
Take A Spanning Tree T (In Red). Then T Is Contractible (why?)

Collapse T To A Point. Then We Are Left With $X/T =$  $= S^1 \vee S^1$

Attaching Spaces This Is A Generalization Of The Mapping Cylinder. Suppose $A \subset X_1$ And $f: A \to X_0$ Is Continuous. Form The Quotient Space

$$X_0 \cup_f X_1 = X_0 \amalg X_1 / a \sim f(a),\ a \in A$$

We Call This Space X_0 with X_1 Attached Along A Via f. In The Special Case $(X_1, A) = (D^n, S^{n-1})$ We Are Attaching An n-Cell To X_0 Via $f: S^{n-1} \to X_0$.



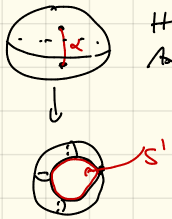
Prop: If (X_1, A) is a CW-pair and two maps $f, g: A \to X_0$ are homotopic, then $X_0 \cup_f X_1 \simeq X_0 \cup_g X_1$.

Pf: Exercise

eg: Consider S^2 and let A = 2 points on S^2.

Claim, $S^2/A \simeq S^1 \vee S^2$

Pf:



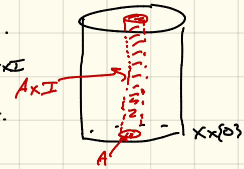
Here's a way to get S^2/A: Take an arc α joining the two points and wrap it around a circle to identify the endpoints.

Since α is contractible, this attaching map f is homotopic to a constant map g. But attaching S^2 to S^1 via g just gives $S^1 \vee S^2$.

Homotopy Extension Property

Q: Suppose $A \subset X$, $f_0: X \to Y$ and $f_t: A \to Y$ is a homotopy of $f_0|_A$. Can we always extend f_t to a homotopy on X $\tilde{f}_t: X \to Y$?

If so: We say that (X, A) has the **Homotopy Extension Property.** (HEP)



Prop: (X, A) has HEP $\iff$ $X \times \{0\} \cup A \times I$ is a retract of $X \times I$.

Pf: $(\Rightarrow)$ The HEP implies that the identity map $X \times \{0\} \cup A \times I \to X \times \{0\} \cup A \times I$ extends to a map $X \times I \to X \times \{0\} \cup A \times I$; so this subspace is a retract.

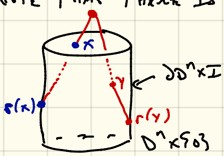
$(\Leftarrow)$ Assume X Hausdorff (not necessary, but it simplifies matters).

First: A is closed: Let $r: X \times I \to X \times I$ be a retraction. The image of r is $\{z \in X \times I \mid r(z) = z\}$ and this is closed if X is Hausdorff. So $X \times \{0\} \cup A \times I$ is closed in $X \times I$ + so A is closed in X.

Now, any two maps $X \times \{0\} \to Y$ + $A \times I \to Y$ that agree on $A \times \{0\}$ combine to give a map $X \times \{0\} \cup A \times I \to Y$ which is continuous. Composing this with the retraction $X \times I \to X \times \{0\} \cup A \times I$ gives an extension $X \times I \to Y$ $\Rightarrow$ (X, A) has HEP.//

Prop: If (X, A) is a CW-pair, then $X \times \{0\} \cup A \times I$ is a deformation retract of $X \times I$ and so (X, A) has HEP.

Pf: Note that there is a retraction $r: D^n \times I \to D^n \times \{0\} \cup \partial D^n \times I$



Set $r_t = tr + (1-t)\,\mathrm{id}$. This is a deformation retraction

$r_t: D^n \times I \to D^n \times \{0\} \cup \partial D^n \times I$

This gives rise to a deformation retraction $X^n \times I \to X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times I$

Perform this on the interval $\left[\frac{1}{2^{n}}, \frac{1}{2^{n-1}}\right]$ + then concatenate to get a def. retraction $X \times I \to X \times \{0\} \cup A \times I$. (Weak topology guarantees continuity)

Cor: If (X,A) has HEP and $A \simeq *$, then $q: X \to X/A$ is a homotopy equiv.

Pf: Let $f_t: X \to X$ be a homotopy extending a contraction of A with $f_0 = \mathrm{Id}_X$ and $f_t(A) \subset A$ for all t. The composition $q f_t: X \to X/A$ sends A to a point and so factors as a composition

$$\begin{tikzcd} X \arrow[r, "q"] \arrow[dr, "qf_t"'] & X/A \arrow[d, dotted, "\exists!\, \bar{f}_t"] \\ & X/A \end{tikzcd}$$ (UMP of quotients)

When $t=1$, $f_1(A)$ is a point. So f_1 induces a map $g: X/A \to X$ with $gq = f_1$ (UMP)

But then we have $qg = \bar{f}_1$ since $qg(\bar{x}) = qgq(x) = q f_1(x) = \bar{f}_1 q(x) = \bar{f}_1(\bar{x})$

$$\begin{tikzcd} X \arrow[r, "f_1"] \arrow[d, "q"'] & X \arrow[d, "q"] \\ X/A \arrow[ur, "\exists!\, g"] \arrow[r, "\bar{f}_1"'] & X/A \end{tikzcd}$$

So g and q are inverse homotopy equivalences:

$gq = f_1 \simeq f_0 = \mathrm{Id}_X$

$qg = \bar{f}_1 \simeq \bar{f}_0 = \mathrm{Id}_{X/A}$. //