Homework # 4 Solutions

Problem 4, p. 177 Let $R$ be the region $0 \leq x \leq \pi$, $0 \leq y \leq 1$. Show that the modulus of $f(z) = \sin z$ has maximum value in $R$ at $z = (\pi/2) + 1$.

By the maximum modulus principle, we know the maximum occurs on the boundary of $R$. Note that $|f(z)|^2 = \sin^2 x + \sinh^2 y$.

Finding the maximum of this function is equivalent to finding the maximum of $|f(z)|$ and since it is a sum of two squares we may find that by maximizing each piece. Note that $\sin^2 x$ attains its maximum when $x = \pi/2$ and $\sinh^2 y$ is maximized when $y = 1$. It follows that we maximize $|f(z)|$ when $z = (\pi/2) + i$.

Problem 2b, p. 195

$$e^z = e \cdot e^{z-1} = e \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}$$

Problem 10b, p. 195

$$\frac{\sin(z^2)}{z^4} = \frac{1}{z^4} \sum_{n=0}^{\infty} (-1)^n \frac{(z^2)^{2n+1}}{(2n+1)!} = \frac{1}{z^2} - \frac{z^2}{3!} + \frac{z^6}{5!} - \cdots$$

Problem 11, p. 195 When $0 < |z| < 4$,

$$\frac{1}{4z - z^2} = \frac{1}{4z} \frac{1}{1 - (z/4)} = \frac{1}{4z} \sum_{n=0}^{\infty} \frac{z^n}{4^n} = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}$$

Problem 2, p. 205 Find a representation for $f(z) = \frac{1}{1+z}$ in negative powers of $z$ valid for $1 < |z| < \infty$. 
Note that $f(z) = \frac{1}{z} \frac{1}{1 + (1/z)}$. When $|z| > 1$ we have

\[
f(z) = \frac{1}{z} \frac{1}{1 + (1/z)} = \frac{1}{z} \sum_{n=0}^{\infty} (-1)^n \frac{1}{z^n} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{z^{n+1}}.
\]

**Problem 5b, p. 205** Find the Laurent expansion in powers of $z$ for $f(z) = \frac{-1}{(z - 1)(z - 2)}$ in the annulus $1 < |z| < 2$.

Note that

\[
f(z) = \frac{1}{z - 1} - \frac{1}{z - 2}.
\]

The second factor is analytic inside $|z| = 2$ so we have

\[
- \frac{1}{z - 2} = \frac{1}{2} \frac{1}{1 - (z/2)} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{z^n}{2^n}.
\]

The first factor will give negative powers of $z$ outside $|z| = 1$. Note that if $|z| > 1$,

\[
\frac{1}{(1/z) - 1} = - \frac{1}{1 - (1/z)} = - \sum_{n=0}^{\infty} \frac{1}{z^n}.
\]

On the other hand, we have

\[
\frac{1}{(1/z) - 1} = \frac{1}{(1 - z)/z} = \frac{z}{1 - z} = - \frac{z}{z - 1}.
\]

We therefore see that

\[
\frac{1}{z - 1} = - \frac{1}{z} \left( - \sum_{n=0}^{\infty} \frac{1}{z^n} \right) = \sum_{n=1}^{\infty} \frac{1}{z^n},
\]

and so

\[
f(z) = \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} + \sum_{n=1}^{\infty} \frac{1}{z^n}.
\]
**Problem 2. p. 218** Substitute $1/(1-z)$ for $z$ in the expansion for $1/(1-z)^2$:

$$
\frac{1}{(1-(1/(1-z)))^2} = \sum_{n=0}^{\infty} \frac{n+1}{(1-z)^n}
$$

$$
= \sum_{n=0}^{\infty} (-1)^n \frac{n+1}{(z-1)^n}.
$$

But note that

$$
\frac{1}{(1-(1/(1-z)))^2} = \frac{1}{(-z/(z-1))^2}
$$

$$
= \frac{(z-1)^2}{z^2},
$$

and so

$$
\frac{1}{z^2} = \frac{1}{(z-1)^2} \sum_{n=0}^{\infty} (-1)^n \frac{n+1}{(z-1)^n}
$$

$$
= \sum_{n=2}^{\infty} (-1)^n \frac{n-1}{(z-1)^n}.
$$