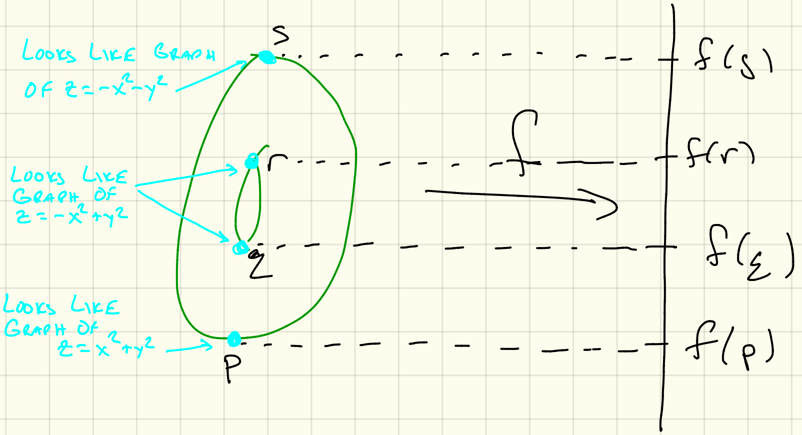
MORSE THEORY

FALL 2018

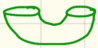
Introduction, Motivation, Basic Definitions

Every Discussion Of Morse Theory Must Contain This Picture:



The Function f Is The "Height" Function. Think Of It As Measuring The Height Of A Point Above Some Plane Below The Torus. Note The Following:

Denote By M^a The Sublevel Set $M^a = \{x \in M \mid f(x) \le a\}$. Observe:

- If $a < f(p)$, $M^a = \emptyset$
- If $f(p) < a < f(q)$ Then M^a Looks Like  And This Is Homeomorphic To A 2-Cell
- If $f(q) < a < f(r)$, Then M^a Looks Like  And This Is Homeomorphic To A Cylinder
- If $f(r) < a < f(s)$, Then M^a Looks Like  And This Is A Surface With Boundary A Circle
- If $f(s) < a$, Then $M^a = M$.

We Will Usually Be Concerned With Homotopy Type And For That We Need The Definition

Attaching Cells If X Is Any Topological Space And $e^k = \{x \in \mathbb{R}^k \mid \|x\| \le 1\}$, Denote The Boundary ∂e^k By S^{k-1}. If $g: S^{k-1} \to X$ Is Continuous Then We Define $X \cup_g e^k$ To Be The Quotient Space

$$X \cup_g e^k = X \amalg e^k / \{x \sim g(x) \text{ for } x \in S^{k-1}\}$$

Our Goal Is To Generalize What Happens With The Height Function On The Torus To An Arbitrary Smooth Manifold.

Smooth Manifolds

"Smooth" Means Differentiable Of Class C^∞. The Tangent Space Of M At p Is Denoted By TM_p. If $g: M \to N$ Is A Smooth Map, Then The Induced Map On Tangent Spaces Is $g_*: TM_p \to TN_{g(p)}$.

Now Suppose $f: M \to \mathbb{R}$ Is Smooth. A Critical Point Of f Is A Point $p \in M$ Where $f_*: TM_p \to T\mathbb{R}_{f(p)}$ Is The Zero Map. In Local Coordinates At p: $(x_1, x_2, \dots, x_n) \Rightarrow \frac{\partial f}{\partial x_i}(p) = 0,\ i = 1, \dots, n$.

RECALL THE IMPLICIT FUNCTION THM: IF a IS NOT A CRITICAL VALUE OF $f: M \to \mathbb{R}$, THEN THE SUBLEVEL SET M^a IS A SMOOTH MANIFOLD WITH BOUNDARY AND $f^{-1}(a)$ IS A SMOOTH MANIFOLD.

A CRITICAL POINT p IS NONDEGENERATE IF THE MATRIX

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p) \right)$$

IS NONSINGULAR.

FACT: THE INVERTIBILITY OF THIS MATRIX DOES NOT DEPEND ON THE COORDINATE SYSTEM (EXERCISE)

HERE'S A BETTER WAY: IF p IS A CRITICAL POINT OF f, DEFINE THE HESSIAN

$$f_{**}: TM_p \times TM_p \longrightarrow \mathbb{R}$$

AS FOLLOWS. IF $v, w \in T_pM$, THEN WE MAY EXTEND THEM TO VECTOR FIELDS $\tilde{v}, \tilde{w}$ IN A NBHD OF p. DEFINE f_{**} BY $f_{**}(v,w) = \tilde{v}_p(\tilde{w}(f))$ [NOTE: THIS IS JUST TAKING THE DIRECTIONAL DERIVATIVE.]

f_{**} IS SYMMETRIC: $\tilde{v}_p(\tilde{w}(f)) - \tilde{w}_p(\tilde{v}(f)) = [\tilde{v}, \tilde{w}]_p(f) = 0$. HERE $[\tilde{v}, \tilde{w}]$ IS THE POISSON BRACKET AND $[\tilde{v}, \tilde{w}]_p(f) = 0$ SINCE p IS A CRITICAL PT OF f.

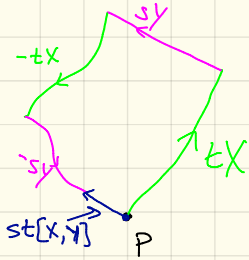
f_{**} IS WELL-DEFINED SINCE $\tilde{v}_p(\tilde{w}(f))$ IS INDEPENDENT OF THE EXTENSION $\tilde{v}$ AND SIMILARLY FOR $\tilde{w}$. SO WE MAY CHOOSE ANY COORDINATE SYSTEM WE LIKE & GET THE SAME RESULT.

IF $v = \sum a_i \frac{\partial}{\partial x_i}\Big|_p$ AND $w = \sum b_j \frac{\partial}{\partial x_j}\Big|_p$ THEN WE CAN TAKE $\tilde{v} = \sum a_i \frac{\partial}{\partial x_i}$ AND $\tilde{w} = \sum b_j \frac{\partial}{\partial x_j}$

THEN
$$f_{**}(v,w) = v(\tilde{w}(f))(p) = v\left(\sum b_j \frac{\partial f}{\partial x_j}\right)(p) = \sum_{i,j} a_i b_j \frac{\partial^2 f}{\partial x_i \partial x_j}(p)$$

ASIDE: THE POISSON BRACKET OF TWO VECTOR FIELDS X AND Y MAY BE VISUALIZED AS FOLLOWS:

FLOW ALONG X FOR t, THEN ALONG Y FOR s, THEN ALONG $-X$ FOR t, THEN $-Y$ FOR s. YOU WON'T BE BACK WHERE YOU STARTED IN GENERAL. $[X,Y]$ MEASURES THIS.



DEF: THE INDEX OF A BILINEAR OPERATOR H ON A VECTOR SPACE V IS THE MAXIMAL DIMENSION OF A SUBSPACE W ON WHICH H IS NEGATIVE DEFINITE (i.e., $H(v,w) < 0$ FOR ALL $v, w \in W$).

DEF: THE NULLITY OF H IS THE DIMENSION OF THE NULL SPACE $N = \{v \in V \mid H(v,w) = 0 \ \forall w \in V\}$.

NOTE: p IS A NONDEGENERATE CRITICAL POINT OF $f \iff$ THE NULLITY OF f_{**} ON TM_p IS 0.

DEF: THE INDEX OF f AT p IS THE INDEX OF f_{**} ON TM_p.

LEMMA: LET f BE A C^∞ FUNCTION IN A CONVEX NBHD OF 0 IN $\mathbb{R}^n$ WITH $f(0) = 0$. THEN
$f(x_1, \dots, x_n) = \sum_{i=1}^n x_i g_i(x_1, \dots, x_n)$ FOR SOME C^∞ FUNCTIONS g_i ON THE NBHD WITH $g_i(0) = \frac{\partial f}{\partial x_i}(0)$.

PROOF: SINCE $f(x_1, \dots, x_n) = \int_0^1 \frac{df(tx_1, \dots, tx_n)}{dt}\, dt = \int_0^1 \sum_{i=1}^n \frac{\partial f}{\partial x_i}(tx_1, \dots, tx_n) \cdot x_i \, dt$, WE CAN TAKE
$g_i(x_1, \dots, x_n) = \int_0^1 \frac{\partial f}{\partial x_i}(tx_1, \dots, tx_n)\, dt$. (OR USE TAYLOR'S THM.)

Morse Lemma: Let p Be A Nondegenerate Critical Point Of f. Then There Is A Local Coordinate System $(y_1, \dots, y_n)$ In A Nbhd U Of p With $y_i(p) = 0$ For All i And Such That

$$f(y_1, \dots, y_n) = f(p) - y_1^2 - y_2^2 - \dots - y_\lambda^2 + y_{\lambda+1}^2 + \dots + y_n^2$$

Throughout U, Where λ Is The Index Of f At p.

Proof: We First Show That If Such An Expression Exists, Then λ Must Be The Index Of f At p. We May Assume $f(p) = 0$. Suppose We Have Coordinates $(z_1, \dots, z_n)$ At p With $f(\vec{z}) = -z_1^2 - z_2^2 - \dots - z_\lambda^2 + z_{\lambda+1}^2 + \dots + z_n^2$, Then We Have

$$\frac{\partial^2 f}{\partial z_i \partial z_j}(p) = \begin{cases} -2 & \text{If } i = j \le \lambda \\ 2 & \text{If } i = j > \lambda \\ 0 & \text{Otherwise} \end{cases}$$

And So The Matrix Representing f_{**} W.R.T. The Basis $\left.\frac{\partial}{\partial z_1}\right|_p, \dots, \left.\frac{\partial}{\partial z_n}\right|_p$ Is

$$\begin{pmatrix} -2 & & & & & \\ & \ddots & & & 0 & \\ & & -2 & & & \\ & & & 2 & & \\ & 0 & & & \ddots & \\ & & & & & 2 \end{pmatrix}$$

So There Is A Subspace Of TM_p Of Dimension λ Where f_{**} Is Negative Definite And A Subspace V Of Dimension $n - \lambda$ Where f_{**} Is Positive Definite. If There Were A Subspace Of TM_p Of Dimension Greater Than λ On Which f_{**} Were Negative Definite Then It Would Intersect V, Which Is Impossible. So λ Is The Index Of f At p.

Now, Choose A Local Coordinate System $(x_1, \dots, x_n)$ At p. We May Write

$$f(x_1, \dots, x_n) = \sum_{i=1}^n x_i g_i(x_1, \dots, x_n)$$

With $\frac{\partial f}{\partial x_i}(p) = g_i(0, \dots, 0)$. But p Is Critical; So $g_i(0, \dots, 0) = 0$, $i = 1, \dots, n$.

Applying The Lemma Again, We Find Smooth Functions $h_{i1}(x_1, \dots, x_n), \dots, h_{in}(x_1, \dots, x_n)$ With

$$g_i(x_1, \dots, x_n) = \sum_{j=1}^n x_j h_{ij}(x_1, \dots, x_n)$$

Putting This Together, We Have

$$f(x_1, \dots, x_n) = \sum_{i,j=1}^n x_i x_j h_{ij}(x_1, \dots, x_n)$$

We May Symmetrize This By Setting $H_{ij}(x_1, \dots, x_n) = (h_{ij} + h_{ji})/2$, And Then

$$f(x_1, \dots, x_n) = \sum_{i,j=1}^n x_i x_j H_{ij}(x_1, \dots, x_n)$$

This Is Called A <u>Quadratic Form Representation</u> Of f. The Form We Seek Is Another Such Representation. We Now Change Coordinates To Get It.

Note That $\frac{\partial^2 f}{\partial x_i \partial x_j}(0, \dots, 0) = 2 H_{ij}(0, \dots, 0)$. Since p Is A Nondegenerate Critical Point We May

May Assume That $\frac{\partial^2 f}{\partial x_1^2}(0, \dots, 0) \neq 0$ (Use A Suitable Linear Change Of Coordinates).

IT FOLLOWS THAT $H_{11}(0,\dots,0) \neq 0$ AND SINCE H_{11} IS CONTINUOUS, $H_{11} \neq 0$ IN A NBHD OF THE ORIGIN.

NOW INTRODUCE A NEW COORDINATE SYSTEM $(X_1, x_2, \dots, x_n)$ WHERE

$$X_1 = \sqrt{|H_{11}|}\left(x_1 + \sum_{i=2}^{n} x_i \frac{H_{1i}}{H_{11}}\right).$$

(EXERCISE: THE JACOBIAN OF THIS TRANSFORMATION IS NONZERO.)

NOTE THAT

$$X_1^2 = |H_{11}|\left(x_1 + \sum_{i=2}^{n} x_i \frac{H_{1i}}{H_{11}}\right)^2 = \begin{cases} H_{11}x_1^2 + 2\sum_{i=2}^{n} x_1 x_i H_{1i} + \left(\sum_{i=2}^{n} x_i H_{1i}\right)^2 / H_{11} & (H_{11} > 0) \\ -H_{11}x_1^2 - 2\sum_{i=2}^{n} x_1 x_i H_{1i} - \left(\sum_{i=2}^{n} x_i H_{1i}\right)^2 / H_{11} & (H_{11} < 0) \end{cases}$$

SO IN THESE COORDINATES WE HAVE

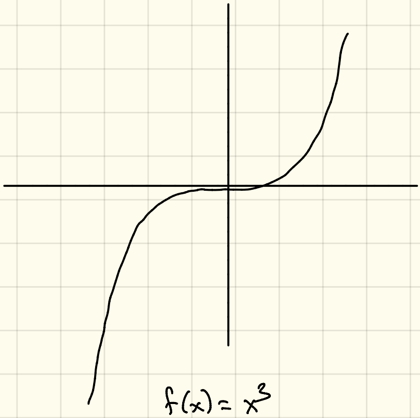
$$f(X_1, x_2, \dots, x_n) = \begin{cases} X_1^2 + \sum_{i,j=2}^{n} x_i x_j H_{ij} - \left(\sum_{i=2}^{n} x_i H_{1i}\right)^2 / H_{11} & (H_{11} > 0) \\ -X_1^2 + \sum_{i,j=2}^{n} x_i x_j H_{ij} - \left(\sum_{i=2}^{n} x_i H_{1i}\right)^2 / H_{11} & (H_{11} < 0) \end{cases}$$

THE SECOND TERM AND ALL SUBSEQUENT TERMS INVOLVE ONLY $x_2, \dots, x_n$. PROCEED BY INDUCTION TO FINISH. //

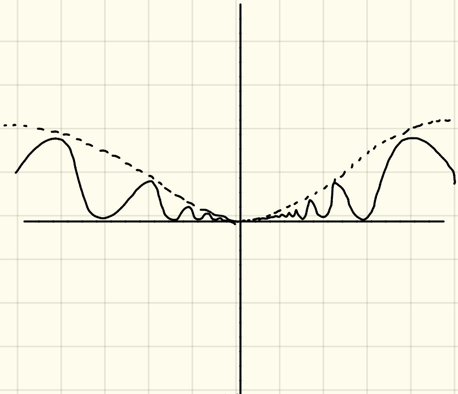
COROLLARY: NONDEGENERATE CRITICAL POINTS ARE ISOLATED. //

COROLLARY: IF M IS COMPACT, A SMOOTH FUNCTION HAS ONLY FINITELY MANY NONDEG. CRITICAL PTS. //

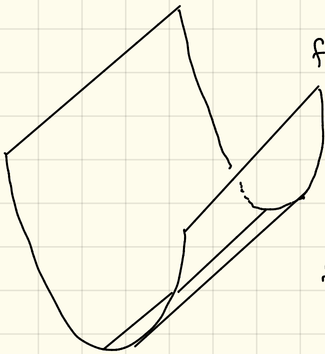
EXAMPLES OF DEGENERATE CRITICAL POINTS



$f(x) = x^3$



$f(x) = e^{-1/x^2}\sin^2(1/x)$



$f(x,y) = x^2$ CRITICAL POINTS $= \{(0,y)\}$ WHICH IS A SUBMANIFOLD OF $\mathbb{R}^2$

Def: A smooth function $f: M \to \mathbb{R}$ is a **Morse function** if it has only nondegenerate critical points.

Do these always exist? **Yes!** In abundance, in fact.

Thm: Let M be a closed manifold and let $g: M \to \mathbb{R}$ be a smooth function. Then there exists a Morse function $f: M \to \mathbb{R}$ arbitrarily close to g.

First, a couple of lemmas.

Lemma: Let U be an open set in $\mathbb{R}^n$ and suppose $f: U \to \mathbb{R}$ is smooth. Then for some real numbers $a_1, \dots, a_n$, the function $f(x_1, \dots, x_n) - (a_1 x_1 + a_2 x_2 + \dots + a_n x_n)$ is a Morse function on U. Moreover, we may choose $a_1, \dots, a_n$ to have arbitrarily small absolute values.

Proof: Define a map $h: U \to \mathbb{R}^n$ by $h = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}$ (i.e. the ith component of h is the partial of f wrt x_i. The Jacobian of h is

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \frac{\partial^2 f}{\partial x_i \partial x_j} & \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

Thus p is a critical point of $h: U \to \mathbb{R}^n \iff$ the determinant of this matrix is 0.

Choose a point $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^n$ which is not a critical value of h. These exist in abundance by Sard's Thm and in fact we may choose the point to be as close to $0 \in \mathbb{R}^n$ as we like. I claim that $\tilde{f}(x_1, \dots, x_n) = f(x_1, \dots, x_n) - (a_1 x_1 + \dots + a_n x_n)$ is a Morse function. If p is a critical point of $\tilde{f}$, then since

$\frac{\partial \tilde{f}}{\partial x_i}(p) = \frac{\partial f}{\partial x_i}(p) - a_i = 0$, $i = 1, \dots, n$, we have $h(p) = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$. But this is not a critical value of h and so p is not a critical point of h. So the determinant of the matrix above is nonzero. But this matrix represents f_{**} and since f and $\tilde{f}$ differ by a linear function, $f_{**} = \tilde{f}_{**}$. Thus, p is a nondegenerate critical point of $\tilde{f}$. //

Def: Suppose K is a compact set in $\mathbb{R}^n$ and $\varepsilon > 0$. We say that f is a (C^2, ε)-approximation of g in K if the following hold at each $p \in K$:

$$|f(p) - g(p)| < \varepsilon$$

$$\left| \frac{\partial f}{\partial x_i}(p) - \frac{\partial g}{\partial x_i}(p) \right| < \varepsilon, \quad i = 1, \dots, n$$

$$\left| \frac{\partial^2 f}{\partial x_i \partial x_j}(p) - \frac{\partial^2 g}{\partial x_i \partial x_j}(p) \right| < \varepsilon, \quad i, j = 1, \dots, n$$

We then define this on a compact manifold M by covering M with a finite collection of coordinate nbhds $U_1, \dots, U_r$ containing compact sets $K_1, \dots, K_r$ which also cover M: $M = K_1 \cup \dots \cup K_r$.