HOMEWORK #1

1. LET T^n BE THE n-DIMENSIONAL TORUS $S^1 \times S^1 \times \cdots \times S^1$. LET $R > 1$ AND DEFINE $f: T^n \to \mathbb{R}$ BY $f(\theta_1, \ldots, \theta_n) = (R + \cos\theta_1)(R + \cos\theta_2)\cdots(R + \cos\theta_n)$. SHOW THAT f IS A MORSE FUNCTION AND CALCULATE THE INDEX OF EACH CRITICAL POINT.

2. LET p AND q BE POINTS IN $\mathbb{R}^2$. IF $x, y \in \mathbb{R}^2$ DENOTE THE DISTANCE FROM x TO y BY $d(x,y)$. DEFINE $f: \mathbb{R}^2 \to \mathbb{R}$ BY $f(x) = d^2(x,p)\, d^2(x,q)$. SHOW THAT f IS A MORSE FUNCTION AND CALCULATE THE INDEX OF EACH CRITICAL POINT. BONUS: WHAT ABOUT THE ANALOGOUS FUNCTION DEFINED USING 3 DISTINCT NONCOLLINEAR POINTS $p, q, r \in \mathbb{R}^2$?

3. RECALL THE MORSE FUNCTION $f: SO(3) \to \mathbb{R}$ GIVEN BY $f((x_{ij})) = c_1 x_{11} + c_2 x_{22} + c_3 x_{33}$ WHERE $1 < c_1 < c_2 < c_3$ ARE FIXED. CONSTRUCT A DIFFEOMORPHISM $\varphi: \mathbb{R}P^3 \to SO(3)$ AND CONSIDER $f \circ \varphi: \mathbb{R}P^3 \to \mathbb{R}$. FIND THE CRITICAL POINTS OF THIS MAP AND COMPUTE THEIR INDICES. (HINT: THERE ARE MANY WAYS YOU MIGHT CONSTRUCT SUCH A φ, BUT I SUGGEST THE FOLLOWING. THINK OF $\mathbb{R}P^3$ AS $D^3/\sim$ WHERE D^3 IS THE UNIT BALL IN $\mathbb{R}^3$ AND $x \sim -x \iff x \in S^2$. POINTS IN D^3 HAVE THE FORM tx FOR $x \in S^2$, $-1 \le t \le 1$; THIS SHOULD SUGGEST A MAP TO $SO(3)$, THE GROUP OF ROTATIONS OF $\mathbb{R}^3$.)