1. Suppose $M$ is a compact, orientable smooth 3-manifold whose integral homology is isomorphic to the homology of $S^3$. Let $f : M \to \mathbb{R}$ be a Morse function.
   (a) Prove that $f$ has an even number of critical points.
   (b) Construct a Morse function on $S^1 \times S^2$ that has exactly 4 critical points.
   (c) Prove that if $H_2(M; \mathbb{Z}) \cong H_2(S^3; \mathbb{Z})$, but $\pi_1(M) \neq \pi_1(S^3)$ (e.g., the Poincaré sphere) then any Morse function on $M$ has at least 6 critical points.

2. Suppose $V$ is an $n$-dimensional real Euclidean space with inner product $\langle \cdot, \cdot \rangle$. Let $A : V \to V$ be a self-adjoint endomorphism. Set $S(V) = \{ v \in V | \|v\| = 1 \}$ and define $f_A : S(V) \to \mathbb{R}$ by $f_A(v) = \langle Av, v \rangle$. For $1 \leq k \leq n$, denote by $G_k(V)$ the Grassmannian of $k$-planes in $V$. Set $\lambda_k = \lambda_k(A) := \min_{E \in G_k(V)} \max_{v \in S(V)} f_A(v)$. Show that $\lambda_1(A) \leq \lambda_2(A) \leq \cdots \leq \lambda_n(A)$ and that any critical value of $f_A$ is equal to one of the $\lambda_k$.

3. Construct a perfect discrete Morse function on the triangulation of $\mathbb{R}P^2$ shown below. (The numbers simply indicate which vertices get identified; they are not function values.)