Homework #2

1. Suppose M is a compact, orientable smooth 3-manifold whose integral homology is isomorphic to the homology of S^3. Let $f: M \to \mathbb{R}$ be a Morse function.
   (a) Prove that f has an even number of critical points.
   (b) Construct a Morse function on $S^1 \times S^2$ that has exactly 4 critical points.
   (c) Prove that if $H_*(M;\mathbb{Z}) \cong H_*(S^3;\mathbb{Z})$, but $\pi_1(M) \neq \{1\}$ (eg, the Poincaré sphere) then any Morse function on M has at least 6 critical points.

2. Suppose V is an n-dimensional real Euclidean space with inner product $\langle \cdot, \cdot \rangle$. Let $A: V \to V$ be a self-adjoint endomorphism. Set $S(V) = \{v \in V \mid \|v\| = 1\}$ and define $f_A: S(V) \to \mathbb{R}$ by $f_A(v) = \langle Av, v \rangle$. For $1 \leq k \leq n$, denote by $G_k(V)$ the Grassmannian of k-planes in V. Set $\lambda_k = \lambda_k(A) := \min_{E \in G_k(V)} \max_{v \in S(V) \cap E} f_A(v)$. Show that

$$\lambda_1(A) \leq \lambda_2(A) \leq \cdots \leq \lambda_n(A)$$ and that any critical value of f_A is equal to one of the λ_k.

3. Construct a perfect discrete Morse function on the triangulation of $\mathbb{R}P^2$ shown below. (The numbers simply indicate which vertices get identified; they are not function values.)