Lemma: Let C be a compact set in an n-manifold M. Suppose that $g: M \to \mathbb{R}$ has no degenerate critical points in C. Then for a sufficiently small $\varepsilon > 0$, any (C^2, ε)-approximation f of g has no degenerate critical points in C.

Proof: Cover C by coordinate nbhds $U_1, \dots, U_r$ containing compact $K_1, \dots, K_r$ which also cover C. Denote coordinates in U_ℓ by $x_1, \dots, x_n$. Note that there are no degenerate critical points of g in $C \cap K_\ell \Leftrightarrow$ the condition

$$\left|\frac{\partial g}{\partial x_1}\right| + \cdots \left|\frac{\partial g}{\partial x_n}\right| + \left|\det\left(\frac{\partial^2 g}{\partial x_i \partial x_j}\right)\right| > 0$$

throughout $C \cap K_\ell$ (Exercise). For a small enough $\varepsilon > 0$, the inequality

$$\left|\frac{\partial f}{\partial x_1}\right| + \cdots + \left|\frac{\partial f}{\partial x_n}\right| + \left|\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)\right| > 0$$

also holds in $C \cap K_\ell$ for any (C^2, ε)-approximation f of g (by definition). Thus, f has no degenerate critical points in $C \cap K_\ell$. Repeating this for all the K_i, we see that f has no degenerate critical points in $C = \bigcup_{\ell=1}^{r} (C \cap K_\ell)$. //

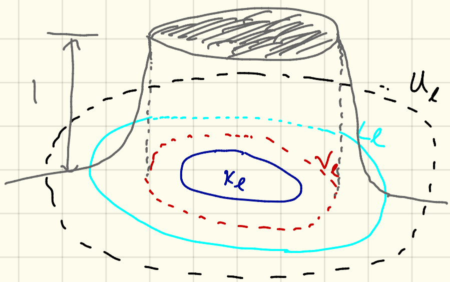
Proof of Existence Thm: Cover M by coordinate nbhds $U_1, \dots, U_r$ containing $K_1, \dots, K_r$. Set f_0 to be the function $g: M \to \mathbb{R}$. We inductively construct functions f_ℓ on M with no degenerate critical points on $K_1 \cup \dots \cup K_\ell$. Denote $K_1 \cup \dots \cup K_\ell$ by C_ℓ. Set $C_0 = \varnothing$. Suppose we have constructed $f_{\ell-1}: M \to \mathbb{R}$ with no degenerate critical point on $C_{\ell-1}$. Consider the coordinate nbhd U_ℓ and its compact subset K_ℓ. Let $(x_1, \dots, x_n)$ be the coordinates in U_ℓ. There exist sufficiently small real numbers $a_1, \dots, a_n$ such that $f_{\ell-1}(x_1, \dots, x_n) - (a_1 x_1 + \dots + a_n x_n)$ is a Morse function on U_ℓ.

Let $h_\ell: U_\ell \to \mathbb{R}$ be a "step function" associated to the pair (U_ℓ, K_ℓ); that is, (i) $0 \le h_\ell \le 1$; (ii) h_ℓ takes the value 1 on a nbhd V_ℓ of K_ℓ; and (iii) h_ℓ is 0 outside a compact set $L_\ell \subset U_\ell$ containing V_ℓ.

Construct f_ℓ by

$$f_\ell = \begin{cases} f_{\ell-1}(x_1, \dots, x_n) - (a_1 x_1 + \dots + a_n x_n)\, h_\ell(x_1, \dots, x_n) & \text{in } U_\ell \\ f_{\ell-1}(x_1, \dots, x_n) & \text{outside } L_\ell \end{cases}$$

These agree on the intersection of the two sets since $h_\ell = 0$ outside L_ℓ.



Note that f_ℓ agrees with the Morse function $f_{\ell-1} - \sum a_i x_i$ in some nbhd of K_ℓ and so f is a Morse function on K_ℓ. We now check that we can take f_ℓ as a (C^2,ε)-approximation of $f_{\ell-1}$. On U_ℓ we have

1. $\left| f_{\ell-1}(p) - f_\ell(p) \right| = \left| (a_1x_1 + \cdots + a_nx_n) \right| h_\ell(p)$
2. $\left| \dfrac{\partial f_{\ell-1}}{\partial x_i}(p) - \dfrac{\partial f_\ell}{\partial x_i}(p) \right| = \left| a_i h_\ell(p) + (a_1x_1 + \cdots + a_nx_n)\dfrac{\partial h_\ell}{\partial x_i}(p) \right|, \quad i = 1, \ldots, n$
3. $\left| \dfrac{\partial^2 f_{\ell-1}}{\partial x_i \partial x_j}(p) - \dfrac{\partial^2 f_\ell}{\partial x_i \partial x_j}(p) \right| = \left| a_i \dfrac{\partial h_\ell}{\partial x_j}(p) + a_j \dfrac{\partial h_i}{\partial x_i}(p) + (a_1x_1 + \cdots + a_nx_n)\dfrac{\partial^2 h_\ell}{\partial x \partial x_j}(p) \right|$

$i, j = 1, \ldots, n$

Since $0 \le h_\ell \le 1$ and is 0 outside a compact set, the absolute values of its first and second derivatives cannot exceed a fixed positive number. So we can make the right hand side of all these arbitrarily small by taking the $|a_i|$ small enough. It follows that f_ℓ can be made (C^2,ε)-close to $f_{\ell-1}$ in K_ℓ.

To approximate f_ℓ on the other K_j, note that the Jacobian between the coordinates on K_ℓ and K_j has all its entries bounded on the compact set $K_j \cap L_\ell$. So by taking the $|a_i|$ small enough, we can make the right hand sides as small as we like on $K_j \cap L_\ell$. But $f_\ell = f_{\ell-1}$ outside L_ℓ and so we have that f_ℓ is a (C^2,ε)-approximation of $f_{\ell-1}$ on C_ℓ.

Now, by the inductive hypothesis, $f_{\ell-1}$ has no degenerate critical point in $C_{\ell-1}$. Since f_ℓ is (C^2,ε)-close to $f_{\ell-1}$ for some small $\varepsilon > 0$, it has no degenerate critical points $C_{\ell-1}$. By construction, f_ℓ has no degenerate critical point in K_ℓ and so f_ℓ is a Morse function on C_ℓ.

Now proceed inductively for $\ell = 1, \ldots, r$. The last f_r is a Morse function on $M = C_r$. Moreover, by taking $\varepsilon > 0$ in each stage of the induction small enough, we can make f_r (C^2,ε')-close to g for any prescribed $\varepsilon' > 0$. //

Examples of Morse Functions

1. The n-Sphere $\quad S^n : \left\{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \;\middle|\; \sum x_i^2 = 1 \right\}$. Consider the height function

$$f : S^n \to \mathbb{R} \qquad f(x_1, \ldots, x_{n+1}) = x_{n+1}$$

This is not a coordinate system on the sphere. But we can parametrize the northern hemisphere via $(x_1, \ldots, x_n, \sqrt{1 - \sum x_i^2})$. In these coordinates, We Have $f(x_1, \dots, x_n) = \sqrt{1 - \sum x_i^2}$. Let's Compute

$$\frac{\partial f}{\partial x_i} = \frac{-x_i}{\sqrt{1 - \sum x_i^2}}$$

It Follows That f Has Exactly One Critical Point In The Northern Hemisphere: $p = (0, 0, \dots, 0, 1)$. The Hessian At p In These Coordinates Is

$$\begin{pmatrix} -1 & & & 0 \\ & -1 & & \\ & & \ddots & \\ 0 & & & -1 \end{pmatrix}$$

And So We See That p Is A Nondegenerate Critical Point Of Index n. Similarly, There Is Another Critical Point In The Southern Hemisphere: $q = (0, \dots, 0, -1)$. This Has Index 0. Note That These Are The Global Maximum And Minimum, Respectively, Of f So It Is No Surprise That These Are Critical Points.

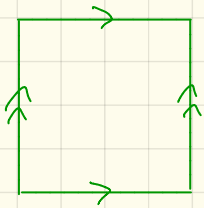
We Have A Converse:

Thm: If $f: M \to \mathbb{R}$ Is A Morse Function On The n-Manifold M With Only Two Critical Points, Then M Is Homeomorphic To S^n.

Proof: Later.//

2. The Torus $T = S^1 \times S^1$ Think Of The Torus As A Quotient Of The Unit Square

Defining A Function On T Is The Same As Defining A Doubly Periodic Function On The Plane. Consider



$f: \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x,y) = \sin^2(\pi x) + \sin^2(\pi y)$

This Descends To A Map $f: T \to \mathbb{R}$. Let's Compute Derivatives:

$\frac{\partial f}{\partial x} = 2\pi \sin(\pi x)\cos(\pi x)$ $\quad \frac{\partial f}{\partial y} = 2\pi \sin(\pi y)\cos(\pi y)$

These Vanish At $x = 0, \frac{1}{2}$, $y = 0, \frac{1}{2}$ And So We Have Four Critical Points:

$$(0,0),\ (\tfrac{1}{2}, 0),\ (0, \tfrac{1}{2}),\ (\tfrac{1}{2}, \tfrac{1}{2})$$

Compute The Hessian: $\frac{\partial^2 f}{\partial x^2} = 2\pi^2(\cos^2(\pi x) - \sin^2(\pi x))$ $\quad \frac{\partial^2 f}{\partial y^2} = 2\pi^2(\cos^2(\pi x) - \sin^2(\pi x))$

$$\frac{\partial^2 f}{\partial x \partial y} = 0$$

So we get $2\pi^2 \begin{pmatrix} \cos(2\pi x) & 0 \\ 0 & \cos(2\pi y) \end{pmatrix}$ which is nonsingular at each critical point.

Compute the indices

C.P	Index	$h(x,y)$
$(0,0)$	0	0
$(\frac{1}{2},0)$	1	1
$(0,\frac{1}{2})$	1	1
$(\frac{1}{2},\frac{1}{2})$	2	2

3. Real Projective Space $\mathbb{R}P^n$ This is the set of lines through the origin in $\mathbb{R}^{n+1}$. It is topologized as a quotient of S^n, which is a double cover via the map $\pi: S^n \to \mathbb{R}P^n$ $\pi(\vec{x})=$ line determined by $\vec{x}$. Note $\pi(\vec{x})=\pi(-\vec{x})$. It follows that $\mathbb{R}P^n$ is compact. We denote points in $\mathbb{R}P^n$ by $[x_1,x_2,\dots,x_n,x_{n+1}]$. Observe that $[x_1,\dots,x_{n+1}]=[y_1,\dots,y_{n+1}]$ $\Leftrightarrow$ there is a nonzero $\alpha\in\mathbb{R}$ with $(y_1,\dots,y_{n+1})=(\alpha x_1,\dots,\alpha x_n)$. Now, choose real numbers $a_1<a_2<\dots<a_n<a_{n+1}$ and define $f:\mathbb{R}P^n\to\mathbb{R}$ by

$$f([x_1,\dots,x_n,x_{n+1}]) = \frac{a_1x_1^2+\dots+a_nx_n^2+a_{n+1}x_{n+1}^2}{x_1^2+\dots+x_n^2+x_{n+1}^2}$$

This is well-defined since it remains unchanged if we scale $(x_1,\dots,x_{n+1})$ by $\alpha\neq 0$. Fix i and let $U_i=\{[x_1,\dots,x_n,x_{n+1}]\in\mathbb{R}P^n \mid x_i\neq 0\}$; this is an open set. Define a coordinate system $(X_1,\dots,X_n)$ on U_i by

$$X_1=\frac{x_1}{x_i},\ \dots,\ X_{i-1}=\frac{x_{i-1}}{x_i},\quad X_i=\frac{x_{i+1}}{x_i},\ \dots,\ X_n=\frac{x_{n+1}}{x_i}$$

In this coordinate system, f has representation

$$f(X_1,\dots,X_n)=\frac{a_1X_1^2+\dots+a_{i-1}X_{i-1}^2+a_i+a_{i+1}X_i^2+\dots+a_{n+1}X_n^2}{X_1^2+\dots+X_{i-1}^2+1+X_i^2+\dots+X_n^2}$$

Compute the derivatives:

$$\frac{\partial f}{\partial X_n}=\frac{2X_n\{(a_{n+1}-a_1)X_1^2+\dots+(a_{n+1}-a_n)X_{n-1}^2+(a_{n+1}-a_i)\}}{(X_1^2+\dots+X_n^2+1)^2}$$

Note that the conditions on the a_i imply that this vanishes $\Leftrightarrow X_n=0$.

Now differentiate $f|_{X_n=0}$ with respect to X_{n-1} to find $\frac{\partial f}{\partial X_{n-1}}=0 \Leftrightarrow X_{n-1}=0$. Repeat to see that the critical points of f on U_i must satisfy $X_i=\dots=X_{n-1}=X_n=0$. Now differentiate wrt X_1 and use the fact that a_1 is minimum to see $\frac{\partial f}{\partial X_1}=0\Leftrightarrow X_1=0$. Repeat to see $X_1=\dots=X_{i-1}=0$. So the only critical point of f in U_i is $(0,0,\dots,0)$. In the notation $[x_1,\dots,x_{n+1}]$ this point is $[0,\dots,0,1,0,\dots,0]$ where the 1 is in ith place.

The Hessian of f at this point is

$$\begin{pmatrix} 2(a_1-a_i) & & & & & \\ & \ddots & & & 0 & \\ & & 2(a_{i-1}-a_i) & & & \\ & & & 2(a_{i+1}-a_i) & & \\ & 0 & & & \ddots & \\ & & & & & 2(a_{n+1}-a_i) \end{pmatrix}$$

It follows that the critical point at the origin of U_i is nondegenerate of index $i-1$. Note that $U_1, \dots, U_{n+1}$ cover $\mathbb{RP}^n$. The Morse function f has $(n+1)$ critical points of indices $0, 1, \dots, n$.

4. Complex Projective Space $\mathbb{CP}^n$ This is a complex manifold of dimension n (so the real dimension is $2n$). Points are denoted the same way: $[z_1, \dots, z_{n+1}]$. Define $f: \mathbb{CP}^n \to \mathbb{R}$ by

$$f(z_1, \dots, z_n, z_{n+1}) = \frac{a_1|z_1|^2 + \dots + a_n|z_n|^2 + a_{n+1}|z_{n+1}|^2}{|z_1|^2 + \dots + |z_n|^2 + |z_{n+1}|^2}$$

where, as before, $a_1 < \dots < a_{n+1}$ are real numbers. Define U_i as in the real case and introduce the analogous complex coordinate system on it. Using a similar argument, we see that the only critical point of f in U_i is $[0, \dots, 0, 1, 0, \dots, 0]$, and its index is $2(i-1)$. The U_i cover $\mathbb{CP}^n$ and f has $n+1$ critical points of indices $0, 2, \dots, 2n$.

5. The Special Orthogonal Group $SO(n)$ This is the group of rotations of $\mathbb{R}^n$. It consists of $n \times n$ matrices A satisfying $AA^T = I$ and $\det A = 1$. Note that $SO(1)$ is the trivial group and $SO(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \middle| \ 0 \le \theta < 2\pi \right\} \simeq S^1$. Since each column of $A \in SO(n)$ is a unit vector, $SO(n)$ is a closed subset of $S^{n-1} \times \dots \times S^{n-1}$ and is therefore compact. The dimension of $SO(n)$ is $\underbrace{(n-1)}_{\text{1st col}} + \underbrace{(n-2)}_{\text{2nd col}} + \dots + 2 + \underbrace{1}_{\text{last col}} = \frac{n(n-1)}{2}$

Fix real numbers $1 < c_1 < \dots < c_n$ and define $f: SO(n) \to \mathbb{R}$ by

$$f(A) = c_1 x_{11} + c_2 x_{22} + \dots + c_n x_{nn} \quad \text{where } A = (x_{ij})$$

Claim: The critical points of f are $\begin{pmatrix} \pm1 & & & 0 \\ & \pm1 & & \\ & & \ddots & \\ 0 & & & \pm1 \end{pmatrix}$

This Is A Tedious Calculation. The Trick Is To Take A Matrix A And Multiply It By The Rotation $B_{jk}(\theta)$, Which Rotates By θ In The j,k-Plane, Then Differentiate WRT θ And Evaluate At 0 To See That $x_{jk} = x_{kj} = 0$ If A Is Critical. Then Since A Is A Rotation It Must Have The Claimed Form. Checking That These Matrices Are In Fact Critical Points Requires Choosing A Basis V_{jk} Of $TSO(n)_A$ And Computing The Derivative Of A In This Direction To See That It Vanishes.

The Calculation Of The Hessian Is More Complicated Of Course, But It Can Be Shown That Each Of These Is A Nondegenerate Critical Point. What Are The Indices? Suppose $A = \mathrm{diag}(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$, $\varepsilon_i = \pm 1$, Is A Critical Point. Suppose The Subscripts i With $\varepsilon_i = 1$ Are $i_1, i_2, \dots, i_r$ In Ascending Order. Then The Index Of The Critical Point A Is

$$(i_1 - 1) + (i_2 - 1) + \dots + (i_r - 1)$$

(The Index Is 0 If All $\varepsilon_i = -1$). Since $\det A = 1$, There Are 2^{n-1} Critical Points (Not 2^n).

Special Case Of $SO(3)$ There Are $2^{3-1} = 4$ Critical Points

$\begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$ Index 0 $\qquad \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$ Index 1 $\qquad \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}$ Index 2

$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$ Index $0+1+2 = 3$

Compare This With $\mathbb{R}P^3$ Which Also Has A Morse Function With 4 Critical Points Of Indices $0, 1, 2, 3$.

One Parameter Families Of Diffeomorphisms This Is A Smooth Map

$$\varphi : \mathbb{R} \times M \to M$$

Such That
1. For Each $t \in \mathbb{R}$, The Map $\varphi_t(x) = \varphi(t,x)$ Is A Diffeomorphism Of M
2. For All $t, s \in \mathbb{R}$, $\varphi_{t+s} = \varphi_t \circ \varphi_s$

Given Such A Family, We Define A Vector Field X On M As Follows. If $f : M \to \mathbb{R}$ Is Smooth Set $X_q(f) = \lim_{h \to 0} \dfrac{f(\varphi_h(q)) - f(q)}{h}$ We Say That X <u>Generates</u> The Group φ.

<u>Lemma</u>: A Smooth Vector Field On M Which Vanishes Outside A Compact Set $K \subset M$ Generates A Unique 1-Parameter Family Of Diffeomorphisms Of M.

<u>Proof</u>: If $t \mapsto c(t)$ Is A Smooth Curve In M, Its Velocity Vector $\frac{dc}{dt} \in TM_{c(t)}$ Is Defined By $\frac{dc}{dt}(f) = \lim_{h \to 0} \dfrac{f(c(t+h)) - f(c(t))}{h}$. Suppose φ Is Generated By A Vector Field X.

For Each Fixed $q \in M$, The Curve $t \mapsto \varphi_t(q)$ Satisfies The Differential Equation

$$\frac{d\varphi_t(q)}{dt} = X_{\varphi_t(q)}$$ With Initial Condition $\varphi_0(q) = q$. This Is True Because

$$\frac{d\varphi_t(q)}{dt}(f) = \lim_{h\to 0} \frac{f(\varphi_{t+h}(q)) - f(\varphi_t(q))}{h} = \lim_{h \to 0} \frac{f(\varphi_h(p)) - f(p)}{h} = X_p(f) \quad (p = \varphi_t(q))$$

But This Diff Eq Has A Unique Solution (Locally) Depending Smoothly On The Initial Condition. Note That, In Local Coordinates $(u_1, \dots, u_n)$ This Equation Has The Form

$$\frac{du_i}{dt} = x_i(u_1, \dots, u_n).$$

So, For Each Point Of M There Is A Nbhd U And $\varepsilon > 0$ So That The Diff Eq

$$\frac{d\varphi_t(q)}{dt} = X_{\varphi_t(q)}, \quad \varphi_0(q) = q$$

Has A Unique Smooth Solution For $q \in U$, $|t| < \varepsilon$.

Cover K By A Finite Number Of Such U. Let ε_0 Be The Smallest Of The ε That Occur. Setting $\varphi_t(q) = q$ For All $q \notin K$, It Follows That The Diff Eq Has A Unique Solution $\varphi_t(q)$ For $|t| < \varepsilon_0$ And All $q \in M$. This Function Is Smooth As A Function Of Both Variables. Also, We Have $\varphi_{t+s} = \varphi_t \circ \varphi_s$ Provided $|t|, |s|, |t+s| < \varepsilon_0$. Thus, Each φ_t Is A Diffeomorphism.

It Remains To Define φ_t For $|t| \geq \varepsilon_0$. Any t Can Be Expressed In The Form $t = k(\varepsilon_0/2) + r$ With k An Integer And $|r| < \varepsilon_0/2$. If $k \geq 0$ Set $\varphi_t = \underbrace{\varphi_{\varepsilon_0/2} \circ \varphi_{\varepsilon_0/2} \circ \cdots \circ \varphi_{\varepsilon_0/2}}_{k \text{ Times}} \circ \varphi_r$

If $k < 0$, Replace $\varphi_{\varepsilon_0/2}$ By $\varphi_{-\varepsilon_0/2}$ And Iterate $-k$ Times. This Defines φ_t For All t And It's Easy To Check $\varphi_{t+s} = \varphi_t \circ \varphi_s$. //