Morse Inequalities

Thm (Weak Morse Inequalities): Suppose $f: M \to \mathbb{R}$ is a Morse function, M compact. For each i, $0 \le i \le n$, denote by c_i the number of critical points of index i and by β_i the ith Betti number ($\beta_i = \operatorname{rank} H_i(M;\mathbb{Z})$). Then for each i

(1) $\beta_i \le c_i$

(2) $\chi(M) = \sum_{i=0}^{n} (-1)^i c_i$

Proof: We know that $M \simeq \underbrace{e^0 \cup \cdots \cup e^0}_{c_0} \cup \underbrace{e^1 \cup \cdots \cup e^1}_{c_1} \cup \cdots \cup \underbrace{e^n \cup \cdots \cup e^n}_{c_n}$. Denote the CW complex on the right by X. Then $\beta_i(M) = \beta_i(X)$ and clearly, $\beta_i(X) \le c_i$. Moreover, it is a standard fact that $\chi(M) = \chi(X) = \sum_{i=0}^{n} (-1)^i \operatorname{rank}(C_i(X)) = \sum_{i=0}^{n} (-1)^i c_i$. //

Thm (Strong Morse Inequalities): For each k, $\beta_k - \beta_{k-1} + \cdots \pm \beta_0 \le c_k - c_{k-1} + \cdots \pm c_0$.

Proof: This is just basic linear algebra, but here's an alternate approach. Consider the Morse polynomial $M_t(f) = \sum_{p \in \mathrm{Crit}(f)} t^{\lambda_p} = \sum_{i=0}^{n} c_i t^i$ and the Poincaré polynomial $P_t(M) = \sum_{i=0}^{n} \beta_i t^i$. How do these differ? Denote by $M_t(f)^a$ the Morse polynomial for the sublevel set M^a. If there are no critical points in $[a,b]$, then $M_t(f)^a = M_t(f)^b$, and $P_t(M^a) = P_t(M^b)$. If there is a single critical point of index λ in $[a,b]$, then $M_t(f)^b - M_t(f)^a = t^\lambda$. What about the Poincaré polynomial? We know $M^b = M^a \cup (D^\lambda \times D^{n-\lambda})$. Consider the attaching sphere $S^{\lambda-1}$. This either bounds a chain in M^a or not. If it does bound a chain, then this chain along with the λ-handle form a new nontrivial cycle of dimension λ. So P_t changes by t^λ and so $\Delta(M_t - P_t) = 0$. If the attaching sphere is a nontrivial cycle in M^a, then the new λ-handle kills it and $\Delta P_t = t^{\lambda-1}$. Thus, $\Delta(M_t - P_t) = t^\lambda + t^{\lambda-1} = t^{\lambda-1}(1+t)$. Proceeding inductively we see

$$M_t(f) - P_t(M) = (1+t)\, Q_t(f)$$

where $Q_t(f)$ is a polynomial with nonnegative integer coefficients. Rewrite this as $Q_t(f) = \frac{M_t - P_t}{1+t} = (M_t - P_t)(1 - t + t^2 - \cdots)$. Now observe that since the coefficients of $Q_t(f)$ are nonnegative we have $c_0 - b_0 \ge 0$; $(b_0 - c_0 + c_1 - b_1) \ge 0$; $(c_2 - b_2 - c_1 + b_1 + c_0 - b_0) \ge 0$, etc. And these are precisely the strong Morse inequalities. //

Handlebodies

Def: A manifold (with boundary) obtained from D^n by attaching handles of various indices in succession: $D^n \cup_{\varphi_1} (D^{\lambda_1} \times D^{n-\lambda_1}) \cup_{\varphi_2} \cdots \cup_{\varphi_r} (D^{\lambda_r} \times D^{n-\lambda_r})$ is called an <u>n-dim'l handlebody</u> denoted $\mathcal{H}(D^n, \varphi_1, \dots, \varphi_r)$.

Thm: If M is a closed manifold and $f: M \to \mathbb{R}$ is a Morse function, then a structure of a handlebody on M is determined by f.

Proof: We've essentially proved this already. If c_i is a critical value of f, we know that $M_{c_i+\varepsilon} = M_{c_i-\varepsilon} \cup_{\varphi_i} (D^{\lambda_i} \times D^{n-\lambda_i})$ where λ_i is the index of the corresponding critical point p_i. Assuming $M_{c_{i-1}+\varepsilon} = \mathcal{H}(D^n, \varphi_1, \dots, \varphi_{i-1})$, since $M_{c_i-\varepsilon}$ is diffeomorphic to $M_{c_{i-1}+\varepsilon}$ (flow along the gradient) we see that $M_{c_i+\varepsilon}$ is a handlebody as well. The induction starts at the minimum critical value c_0 where $M_{c_0+\varepsilon} \approx D^n$ is a trivial handlebody. //

Sliding Handles

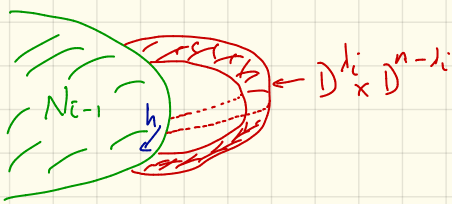
Idea: Alter the attaching maps without changing the diffeomorphism type of the handlebody. Say $p_0, \dots, p_r$ are critical points with $c_0 < c_1 < \dots < c_r$. Then we can write $M = D^n \cup_{\varphi_1} (D^{\lambda_1} \times D^{n-\lambda_1}) \cup_{\varphi_2} \cdots \cup_{\varphi_r} D^n$ (this last disc corresponds to $c_r = \max(f(x))$). Denote by N_j the subhandlebody $\mathcal{H}(D^n, \varphi_1, \dots, \varphi_j)$. The ith handle is attached to N_{i-1} via $\varphi_i : \partial D^{\lambda_i} \times D^{n-\lambda_i} \to \partial N_{i-1}$. "Sliding the handle $D^{\lambda_i} \times D^{n-\lambda_i}$" is to perturb φ_i by an isotopy of ∂N_{i-1}.

Def: Let K be a k-manifold. A family $\{h_t\}_{t \in J}$, J an open interval in $\mathbb{R}$, of diffeos $h_t : K \to K$ is called an isotopy of K if

(1) The open interval J contains $[0,1]$ and $h_t = h_0 = id_K$ when $t \le 0$, and for any $t \ge 1$ $h_t = h_1 = h$, a diffeomorphism of K.

(2) The map $H : K \times J \to K \times J$ defined by $H(x,t) = (h_t(x), t)$ is a diffeomorphism.

Thm (Handle Sliding): Fix i. Given an isotopy $\{h_t\}_{t \in J}$ of ∂N_{i-1}, the attaching map φ_i of the handle can be replaced by $h \circ \varphi_i$ $(h = h_1)$. This replacement does not change the diffeomorphism type of each of the subhandlebodies N_j, $0 \le j \le n$.



Proof: Let $c_j = f(p_j)$. For each j, $N_j = \mathcal{H}(D^n, \varphi_1, \dots, \varphi_j)$ can be identified with $M^{c_j+\varepsilon}$ for small $\varepsilon > 0$. Let's look at the ith handle. Denote the attaching map by $\varphi : \partial D^{\lambda_i} \times D^{n-\lambda_i} \to \partial M^{c_i-\varepsilon}$. Denote by $\Phi : M^{c_{i-1}+\varepsilon} \to M^{c_i-\varepsilon}$ the diffeomorphism induced by choosing a gradient-like vector field X for f. By identifying N_{i-1} with $M^{c_{i-1}+\varepsilon}$, the attaching map φ_i where the ith handle is attached to N_{i-1} is

$$\varphi_i = \Phi^{-1} \circ \varphi : \partial D^{\lambda_i} \times D^{n-\lambda_i} \to \partial N_{i-1}.$$

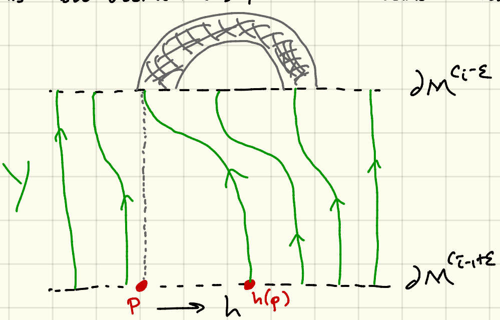
Note that $f^{-1}([c_{i-1}+\varepsilon, c_i-\varepsilon])$ is diffeomorphic to $\partial M^{c_{i-1}+\varepsilon} \times [0,1]$. Moreover, under this identification, for each $p \in \partial M^{c_{i-1}+\varepsilon}$ the interval $\{p\} \times [0,1]$ corresponds to the integral curve of X passing through p. In fact this diffeomorphism may be defined on a larger region since $[c_{i-1}+\varepsilon/2, c_i-\varepsilon/2]$ also does not contain a critical value of f. So for a sufficiently small $\delta > 0$, we can extend this diffeo to a diffeo $f^{-1}([c_{i-1}+\varepsilon/2, c_i-\varepsilon/2]) \cong \partial M^{c_{i-1}+\varepsilon} \times [-\delta, 1+\delta]$. And, $\{p\} \times [-\delta, 1+\delta]$ still corresponds to an integral curve of X.

We may now interpret Φ in terms of the right hand side as a diffeo that stretches $\partial M^{c_{i-1}+\varepsilon} \times [-\delta, 0]$ to $\partial M^{c_{i-1}+\varepsilon} \times [-\delta, 1]$; that is, we may write

$$\Phi : (p,t) \mapsto \left(p, \frac{1+\delta}{\delta} t + 1\right), \quad (p,t) \in \partial M^{c_{i-1}+\varepsilon} \times [-\delta, 0].$$

By identifying N_{i-1} and $M^{c_{i-1}+\varepsilon}$, we are giving an isotopy $\{h_t\}_{t \in J}$ on $\partial M^{c_{i-1}+\varepsilon}$. It follows that $H : \partial M^{c_{i-1}+\varepsilon} \times J \to \partial M^{c_{i-1}+\varepsilon} \times J$, $H(x,t) = (h_t(x), t)$ is a diffeomorphism. Note that H is constant, independent of t for $t \le 0$ and $t \ge 1$. Denote by $\tilde{H}$ the map $\tilde{H}(x,t) = (h_{1-t}(x), t)$. The map $\tilde{H} : \partial M^{c_{i-1}+\varepsilon} \times J \to \partial M^{c_{i-1}+\varepsilon} \times J$ is also a diffeo and is constant for $t \le 0$, $t \ge 1$. Since $J \supset [0,1]$, we can identify J with $(-\delta, 1+\delta)$.

Note that on $\partial M^{c_{i-1}+\varepsilon} \times (-\delta, 1+\delta)$ the vector field X has integral curves $\{p\} \times (-\delta, 1+\delta)$ and so we may identify it with $\frac{\partial}{\partial t}$. Consider $\tilde{H}_*(X)$, a vector field on $\partial M^{c_{i-1}+\varepsilon} \times (-\delta, 1+\delta)$. Since $\tilde{H}$ is constant for $t \le 0$ and $t \ge 1$, $\tilde{H}_*(X) = \frac{\partial}{\partial t}$ $(= X)$ in this range. So if we replace X by $\tilde{H}_*(X)$ in $\partial M^{c_{i-1}+\varepsilon} \times (-\delta, 1+\delta)$, the new vector field extends smoothly to the original X outside $\partial M^{c_{i-1}+\varepsilon} \times (-\delta, 1+\delta)$. Call this new vector field Y. Its integral curves look like this:



The Diffeomorphism $\Phi: M^{c_{i-1}+\varepsilon} \to M^{c_i-\varepsilon}$ Was Determined By Flowing Along The Integral Curves Of X. In The Same Way, Y Determines A Diffeomorphism $\Psi: M^{c_{i-1}+\varepsilon} \to M^{c_i-\varepsilon}$. Restricting Ψ To The Boundary $\partial M^{c_{i-1}+\varepsilon}$, We See That $\Psi: (h(p),0) \mapsto (p,1)$ For Any $p \in \partial M^{c_{i-1}+\varepsilon}$. Thus, In The Handle Decomposition Determined By f And Y, The ith Handle Is Attached To $N_{i-1} = M^{c_{i-1}+\varepsilon}$ By The Map $\Psi^{-1} \circ \varphi = h \circ \varphi_i$. The Structure Of The Handlebodies Up To The $(i-1)$-th Handle Remains Unchanged Since X And Y Agree On Those Handlebodies. It Is Also Clear That The Diffeo Type Of $N_j = M^{c_j+\varepsilon}$ Remains Unchanged For Any j Since The Definition Of $M^{c_j+\varepsilon} = \{p \in M \mid f(p) \le c_j + \varepsilon\}$ Does Not Depend On The Choice Of Gradient-Like Vector Field. //

Thm: Let M Be A Closed n-Manifold And Let $f: M \to \mathbb{R}$ Be A Morse Function. Then f Can Be Perturbed In Such A Way That, After The Perturbation, For Any Critical Points p_i And p_j, $f(p_i) < f(p_j)$ Implies $\text{index}(p_i) \le \text{index}(p_j)$.

This Requires Some Technical Lemmas:

Lemma: (General Position) Let S_1 And S_2 Be Compact Submanifolds Of Dimensions s_1 And s_2 In A k-Manifold K. If $s_1 + s_2 < k$, Then There Is An Isotopy $\{h_t\}_{t \in J}$ Of K Such That $h_0 = id$ And $h_1(S_1) \cap S_2 = \emptyset$.

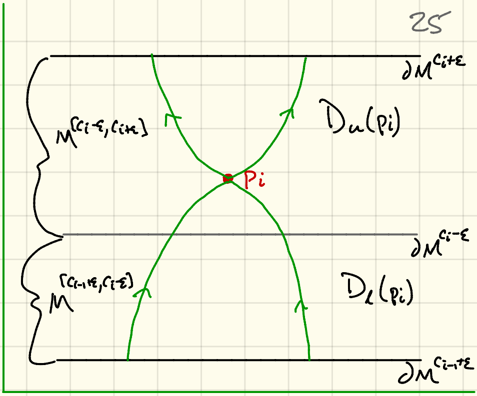
Proof: We Assume That S_1 Has A Tubular Nbhd $U \cong S_1 \times \text{int}(D^{k-s_1})$, Where $S_1 \leftrightarrow S_1 \times \{0\}$ In U. Denote By $\pi: U \to \text{int}(D^{k-s_1})$ The Projection To The Second Factor. By Mapping $S_2 \cap U$ Into $\text{int}(D^{k-s_1})$ Under π, We See $\dim(S_2 \cap U) \le s_2 < k - s_1 = \dim(D^{k-s_1})$. It Follows That $\pi(S_2 \cap U)$ Is Nowhere Dense. Choose p Near 0 In $\text{int}(D^{k-s_1})$ Which Is Not Contained In $\pi(S_2 \cap U)$.

There Is An Isotopy $\{j_t\}_{t \in J}$ Of $\text{int}(D^{k-s_1})$ With The Following Properties:

(i) $j_0 = id$ And $j_1(0) = p$
(ii) For Any t, $j_t = id$ Outside $\frac{1}{2} D^{k-s_1}$ (Exercise)

Now Define An Isotopy $\{h_t\}_{t \in J}$ On U Via $h_t(q,x) = (q, j_t(x))$. This Is The Identity Outside $S_1 \times \frac{1}{2} D^{k-s_1}$ And So May Be Extended To An Isotopy On All Of K. After Moving S_1 By This Isotopy, $h_1(S_1) = S_1 \times \{p\}$ In U. But This Choice Of p Guarantees $S_1 \times \{p\}$ Does Not Intersect $S_2 \cap U$. Thus, $h_1(S_1) \cap S_2 = \emptyset$. //

Def: (Upper And Lower Discs) The Set Of All Points In $M^{[c_{i-1}+\varepsilon,\, c_i-\varepsilon]} = \{p \in M \mid c_{i-1}+\varepsilon \le f(p) \le c_i+\varepsilon\}$ That Converge To The Critical Point p_i Along Integral Curves Of X As $t \to \infty$ Is Called The Lower Disc Associated With p_i, Denoted $D_\ell(p_i)$. The Set Of Points That Converge To p_i Along An Integral Curve As $t \to -\infty$ Is Called The Upper Disc Denoted $D_u(p_i)$. (Note: These Are Sometimes Called The Descending And Ascending Discs, Resp.)



Since A Gradient-Like Vector Field For X Matches The Standard Form In A Nbhd Of p_i We See That $D_\ell(p_i) \cap M^{[c_i-\varepsilon,\, c_i+\varepsilon]} = \{(x_1, \dots, x_n) \mid x_1^2 + \dots + x_{\lambda_i}^2 \le \varepsilon,\ x_{\lambda_i+1} = \dots = x_n = 0\}$ Is A λ_i-Disc. Also, $D_\ell(p_i) \cap M^{[c_{i-1}+\varepsilon,\, c_i-\varepsilon]}$ = Boundary Of This λ_i-Disc $\times [c_{i-1}+\varepsilon, c_i-\varepsilon]$.

Since $D_\ell(p_i)$ Is The Union Of These, We See That $D_\ell(p_i)$ Is Diffeomorphic To A λ_i-Disc. Similarly, $D_u(p_i)$ Is Diffeomorphic To An $(n-\lambda_i)$-Disc. If We Regard $M^{c_i+\varepsilon}$ As $M^{c_{i-1}+\varepsilon}$ With A λ_i-Handle Attached, $D_\ell(p) \leftrightarrow$ Core And $D_u(p_i) \leftrightarrow$ Co-Core. The Boundary $\partial D_\ell(p_i)$ Is Embedded In $\partial M^{c_{i-1}+\varepsilon}$ By The Attaching Map φ_i Of The λ_i-Handle. Moreover, $\partial D_u(p_{i-1})$ Is An Embedded Sphere In $\partial M^{c_{i-1}+\varepsilon}$ As Well.

Lemma: (Separation Of Handles) Fix i. If $\operatorname{index}(p_{i-1}) \ge \operatorname{index}(p_i)$, Then The Gradient-Like Vector Field X Can Be Perturbed To Another Gradient-Like Vector Field Y, In Such A Way That By Replacing The Attaching Map φ_i By ψ_i Using This Perturbation, We Have $\psi_i(\partial D_\ell(p_i)) \cap \partial D_u(p_{i-1}) = \emptyset$, Keeping The Morse Function $f: M \to \mathbb{R}$ Fixed. This Perturbation Does Not Alter The Handles From The 0th To The $(i-1)$th.

Proof: Denote The Indices Of p_i And p_{i-1} By λ_i And λ_{i-1}. We've Assumed $\lambda_{i-1} \ge \lambda_i$. The Dimensions Of $D_u(p_{i-1})$ And $D_\ell(p_i)$ Are $n-\lambda_{i-1}$ And λ_i, Respectively. Note That

$$\dim(\partial D_u(p_{i-1})) + \dim(\partial D_\ell(p_i)) = (n-\lambda_{i-1}-1) + (\lambda_i - 1) = n-2+(\lambda_i-\lambda_{i-1}) < n-1.$$

Since $\dim(\partial M^{c_{i-1}+\varepsilon}) = n-1$, General Position Implies The Existence Of An Isotopy $\{h_t\}_{t\in J}$ Of $\partial M^{c_{i-1}+\varepsilon}$ Separating The Image Of $\varphi_i(\partial D_\ell(p_i))$ From $\partial D_u(p_{i-1})$: $h_1(\varphi_i(\partial D_\ell(p_i))) \cap \partial D_u(p_{i-1}) = \emptyset$. Applying Handle Sliding, The Attaching Map φ_i Can Be Perturbed To $h_1 \circ \varphi_i$. Call This Map ψ_i. Then $\psi_i(\partial D_\ell(p_i)) \cap \partial D_u(p_{i-1}) = \emptyset$. It Is Clear We've Not Changed The Previous Handles.

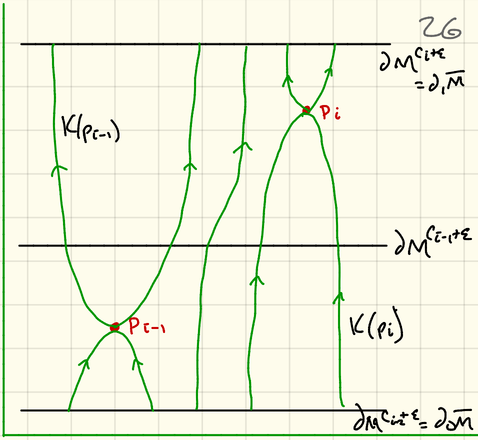
LEMMA: (Moving Critical Values Up And Down)

Consider the set $M^{[c_{i-2}+\varepsilon,\, c_i+\varepsilon]}$ which contains two consecutive critical points p_{i-1} and p_i of $f: M \to \mathbb{R}$. Let $K(p_{i-1})$ be the set of points p that converge to p_{i-1} along integral curves of X as $t \to \infty$ or $t \to -\infty$ (p_{i-1} is considered part of this set). Define $K(p_i)$ similarly. If $K(p_{i-1}) \cap K(p_i) = \emptyset$, then f can be perturbed to another Morse function $g: M \to \mathbb{R}$ such that

(i) $g = f$ outside $M^{[c_{i-2}+\varepsilon,\, c_i+\varepsilon]}$

(ii) The sets of critical points and their indices agree for f and g

(iii) For any given $a, b \in (c_{i-2}+\varepsilon,\, c_i+\varepsilon)$, $g(p_{i-1}) = a$ and $g(p_i) = b$.



Proof: Set $\bar{M} = M^{[c_{i-2}+\varepsilon,\, c_i+\varepsilon]}$, $\partial_0 \bar{M} = f^{-1}(c_{i-2}+\varepsilon)$, $\partial_1 \bar{M} = f^{-1}(c_i+\varepsilon)$. The integral curve of X passing through a point $p \in \bar{M} - (K(p_{i-1}) \cup K(p_i))$ enters through $\partial_0 \bar{M}$ and exits via $\partial_1 \bar{M}$. Choose $h: \partial_0 \bar{M} \to \mathbb{R}$ such that (i) $0 \le h \le 1$; (ii) $h = 0$ on some nbhd of $K(p_{i-1}) \cap \partial_0 \bar{M}$ and takes value 1 on some open nbhd of $K(p_i) \cap \partial_0 \bar{M}$. Now define $\bar{h}: \bar{M} \to \mathbb{R}$ as follows:
If $p \in \bar{M} - (K(p_i) \cup K(p_{i-1}))$, the integral curve for X through p intersects $\partial_0 \bar{M}$ at a unique q. Set $\bar{h}(p) = h(q)$. Set $\bar{h} = 0$ on $K(p_{i-1})$ and $\bar{h} = 1$ on $K(p_i)$. The function $\bar{h}$ is smooth and takes a constant value on each integral curve of X. Now define $G: [c_{i-2}+\varepsilon,\, c_i+\varepsilon] \times [0,1] \to [c_{i-2}+\varepsilon,\, c_i+\varepsilon]$ to be a smooth function satisfying

(a) $G(x,s)$ is strictly increasing as a function of x when s is fixed and $G(x,s)$ increases from $c_{i-2}+\varepsilon$ to $c_i+\varepsilon$ as x increases from $c_{i-2}+\varepsilon$ to $c_i+\varepsilon$.

(b) $G(f(p_{i-1}), 0) = a$, $G(f(p_i), 1) = b$

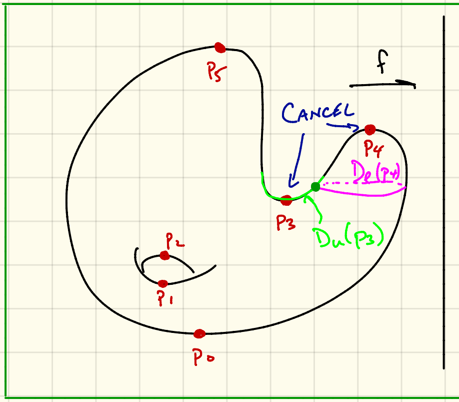
(c) For any s, $G(x,s) = x$ as long as x is in a sufficiently small nbhd of $c_{i-2}+\varepsilon$ or $c_i+\varepsilon$. Furthermore, for x in a nbhd of $f(p_{i-1})$, $\frac{\partial}{\partial x} G(x,0) = 1$ and for x in a nbhd of $f(p_i)$, $\frac{\partial}{\partial x} G(x,1) = 1$.

Define $g: M \to \mathbb{R}$ by $g(p) = G(f(p), \bar{h}(p))$. This is the Morse function we seek. //

Proof of the Theorem: Let f be the given Morse function and let X be a gradient-like vector field for f. Suppose the critical points are $p_0, \dots, p_r$ with corresponding critical values $c_0 < c_1 < \dots < c_r$. We'd be done if the indices were already in ascending order; suppose $\mathrm{index}(p_{i-1}) > \mathrm{index}(p_i)$ for some i. By separation of handles, we may perturb X to Y so that $\partial D_\ell(p_i) \cap \partial D_u(p_{i-1}) = \emptyset$ in $f^{-1}(c_{i-1}+\varepsilon)$. Then $\partial D_\ell(p_i)$ does not converge to p_{i-1} even when it flows downwards along Y, and $\partial D_u(p_{i-1})$ does not converge to p_i even when it flows up along Y.

It Follows That $K(p_{i-1})$ And $K(p_i')$ Are Disjoint In $M^{[c_{i-2}+\varepsilon,\, c_i+\varepsilon]}$. We May Now Apply The Last Lemma. Choose $a>b$ In $(c_{i-2}+\varepsilon, c_i+\varepsilon)$. Then f Can Be Perturbed In $M^{[c_{i-2}+\varepsilon,\, c_i+\varepsilon]}$ To $g: M\to\mathbb{R}$ With $g(p_{i-1})=a>b=g(p_i)$. The Values Of g Remain Unchanged At The Other p_j. Rename p_{i-1} And p_i. We Then Have $g(p_{i-1})<g(p_i)$, $\text{index}(p_{i-1})<\text{index}(p_i)$. Correcting These One At A Time Fixes All The Problems. //

Canceling Handles



Suppose $f: M\to\mathbb{R}$ Has Gradient-Like Vector Field X, Critical Points $p_0,\dots,p_r$ With $c_0<c_1<\dots<c_r$. Fix i And Consider $\overline{M}=M^{[c_{i-2}+\varepsilon,\, c_i+\varepsilon]}$, Which Contains p_{i-1} And p_i.

Thm: (Canceling Handles) Assume The Following:

(i) $\text{index}(p_i)=\text{index}(p_{i-1})+1$

(ii) $\partial D_\ell(p_i)$ And $\partial D_u(p_{i-1})$ Intersect Transversely At A Single Point In The Level Surface $f^{-1}(c_{i-1}+\varepsilon)$.

Then f Can Be Perturbed To Another Morse Function g Such That

(a) g Has No Critical Points In $\text{int}(\overline{M})$

(b) $g=f$ Near $\partial\overline{M}$ And Outside $\overline{M}$.

Proof: Let q_0 Be The Intersection Pt Of $\partial D_\ell(p_i)$ And $\partial D_u(p_{i-1})$ In $f^{-1}(c_{i-1}+\varepsilon)$. Then The Integral Curve $C(t)$ Of X Passing Through q_0 Satisfies $C(t)\to p_i$ As $t\to\infty$ And $C(t)\to p_{i-1}$ As $t\to-\infty$. $C(t)$ Is The Only Integral Curve Of X With This Property.

Set $\text{index}(p_{i-1})=\lambda$. In Local Coordinates Around p_{i-1} X Has The Form
$X=-2x_1\frac{\partial}{\partial x_1}-\dots-2x_\lambda\frac{\partial}{\partial x_\lambda}+2x_{\lambda+1}\frac{\partial}{\partial x_{\lambda+1}}+\dots+2x_n\frac{\partial}{\partial x_n}$. For Convenience, We Interchange x_1 & $x_{\lambda+1}$ So That $X=2x_1\frac{\partial}{\partial x_1}-\dots-2x_\lambda\frac{\partial}{\partial x_\lambda}-2x_{\lambda+1}\frac{\partial}{\partial x_{\lambda+1}}+\dots+2x_n\frac{\partial}{\partial x_n}$.

Similarly, Near p_i, We May Write $X=-2y_1\frac{\partial}{\partial y_1}-\dots-2y_{\lambda+1}\frac{\partial}{\partial y_{\lambda+1}}+2y_{\lambda+2}\frac{\partial}{\partial y_{\lambda+2}}+\dots+2y_n\frac{\partial}{\partial y_n}$.

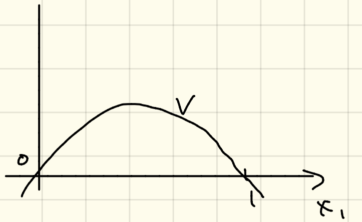
The Idea Is Now To Find A Nbhd Of The Curve $C(t)$ With Coordinates $(x_1,\dots,x_n)$ Such That

(i) In These Coordinates, $p_{i-1}=(0,\dots,0)$.

(ii) In These Coordinates, $p_i=(1,0,\dots,0)$

(iii) In This Nbhd, $X=2v(x_1)\frac{\partial}{\partial x_1}-2x_2\frac{\partial}{\partial x_2}-\dots-2x_{\lambda+1}\frac{\partial}{\partial x_{\lambda+1}}+2x_{\lambda+2}\frac{\partial}{\partial x_{\lambda+2}}+\dots+2x_n\frac{\partial}{\partial x_n}$

Here, $v(x_1)$ Is A Smooth Function On The Interval $-\delta < x_1 < 1+\delta$ With $v(x_1)=x_1$ In A Small Nbhd Of 0 + $v(x_1)=1-x_1$ In A Small Nbhd Of 1. Then X Has The Correct Form Near p_{i-1} And After The Coordinate Change $(y_1,\dots,y_n)=(x_1-1,x_2,\dots,x_n)$ It's Correct Near p_i As Well. The Difficult Part Is To Construct This Nbhd Of $C(t)$, See J. Milnor, Lectures On The h-Cobordism Thm.



Let's Assume We've Done This. We Now Perturb X In This Nbhd U Of $C(t)$ So That $X \neq 0$ Everywhere In U. To Do This, We Construct A Family $\{v_s(x_1)\}_s$ Of Functions Satisfying

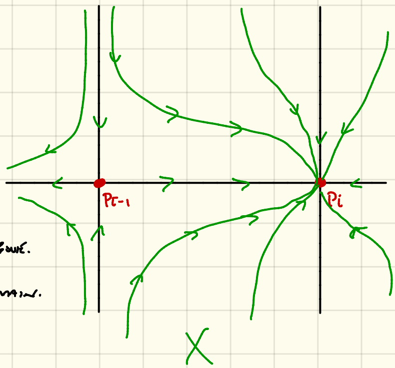
(i) Each $v_s(x_1)$ Is Smooth On $(-\delta, 1+\delta)$

(ii) For Sufficiently Small η, The $v_s(x_1)$ Are Defined As Long As $-\eta < s < 2\eta$.

(iii) For Any $s \geq \eta$, $v_s(x_1) = v(x_1)$, Where $v(x_1)$ Is The Function Above.

(iv) For Any $s \leq 0$, $v_s(x_1) = v_0(x_1)$ And $v_0(x_1) < 0$ For Any x_1 In Domain.

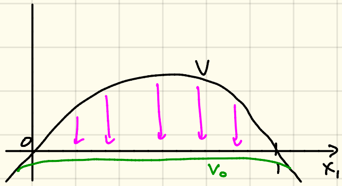
(v) If $x_1 < -\delta/2$ or $x_1 > 1+\delta/2$, Then $v_s(x_1) = v(x_1)$ For All s.



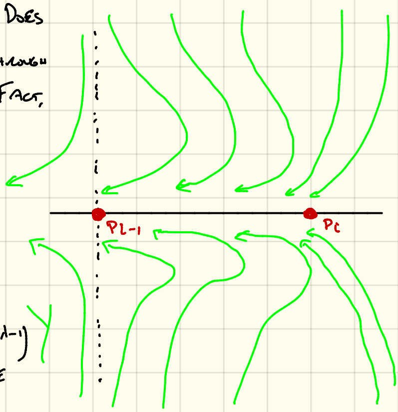
In U Perturb X To The Vector Field $\tilde{X}$:

$$\tilde{X} = 2v_\rho(x_1)\frac{\partial}{\partial x_1} - 2x_2\frac{\partial}{\partial x_2} - \cdots - 2x_{\lambda+1}\frac{\partial}{\partial x_{\lambda+1}} + \cdots + 2x_n\frac{\partial}{\partial x_n}$$

Where $\rho: U \to \mathbb{R}$ Is $\rho = x_2^2 + \cdots + x_n^2$ And $v_\rho(x_1)$ Is Obtained From The Family $\{v_s(x_1)\}$. By (iii) And (v), $\tilde{X} = X$ Away From C. The Second And Later Terms Are Not Zero For Points Not On x_1-Axis.



On The x_1-Axis, $\tilde{X} \neq 0$ Since The First Term Satisfies $2v_0(x_1) < 0$ ($\rho = 0$). Thus $\tilde{X} \neq 0$ In U. Since $\tilde{X} = X$ Away From C It Extends Smoothly To X Outside U Yielding A Vector Field Y On M. X And Y Differ Only In A Small Nbhd Of $C(t)$. In $\bar{M}$, Y Does Not Vanish. Moreover, The Integral Curves Of Y Enter $\bar{M}$ Through $\partial_0\bar{M} = f^{-1}(c_{i-2}+\varepsilon)$ And Exit Via $\partial_1\bar{M} = f^{-1}(c_i+\varepsilon)$. Using This Fact, We Can Define $\bar{f}: \bar{M} \to \mathbb{R}$ That Increases Smoothly Along The Integral Curves Of Y From $c_{i-2}+\varepsilon$ To $c_i+\varepsilon$. Also, $\bar{f}$ Can Be Constructed So That It Matches f On A Nbhd Of The Boundary Of $\bar{M}$. We Then Extend To $g: M \to \mathbb{R}$ And Y Is A Gradient-Like Vector Field For g."



In Terms Of Handles: If $N' = N \cup (D^\lambda \times D^{n-\lambda})$, $N'' = N' \cup (D^{\lambda+1} \times D^{n-\lambda-1})$ And $0 \times \partial D^{n-\lambda}$ And $D^{\lambda+1} \times 0$ Intersect Transversely In A Single Point In $\partial N'$, Then N'' Is Diffeomorphic To N.