Min-Max Theory

How can we find critical points of smooth functions? How many are there? If f is Morse, then the number of critical points is bounded below by $\sum_{j=0}^{n} \beta_j$. If M is compact, there are at least two critical points: the global min + global max.

Min-Max Theory is a mechanism to find saddle-type critical points.

Def: A collection of Min-Max Data for the smooth function $f: M \to \mathbb{R}$ is a pair $(\mathcal{H}, \mathcal{S})$ satisfying the following conditions:

(i) $\mathcal{H}$ is a collection of homeomorphisms of M such that for every regular value a of f there is an $\varepsilon > 0$ and $h \in \mathcal{H}$ such that

$$h(M^{a+\varepsilon}) \subset M^{a-\varepsilon}.$$

(ii) $\mathcal{S}$ is a collection of subsets of M such that

$$h(S) \in \mathcal{S} \quad \forall h \in \mathcal{H}, \forall S \in \mathcal{S}.$$

Thm: (Min-Max Principle) If $(\mathcal{H}, \mathcal{S})$ is a collection of Min-Max Data for $f: M \to \mathbb{R}$, then the real number $c = \inf_{S \in \mathcal{S}} \sup_{x \in S} f(x)$ is a critical value of f.

Proof: Suppose not; that is, assume that c is a regular value. Then there exist $\varepsilon > 0$ and $h \in \mathcal{H}$ so that $h(M^{c+\varepsilon}) \subset M^{c-\varepsilon}$. From the definition of c, we see that there is an $S \in \mathcal{S}$ such that $\sup_{x \in S} f(x) < c + \varepsilon$; that is, $S \subset M^{c+\varepsilon}$. Then $S' = h(S) \in \mathcal{S}$ and $h(S) \subset M^{c-\varepsilon}$. It follows that $\sup_{x \in S'} f(x) \leq c - \varepsilon$ so that $\inf_{S' \in \mathcal{S}} \sup_{x \in S'} f(x) \leq c - \varepsilon$, contrary to the choice of c as a min-max value. //

How to Produce Min-Max Data

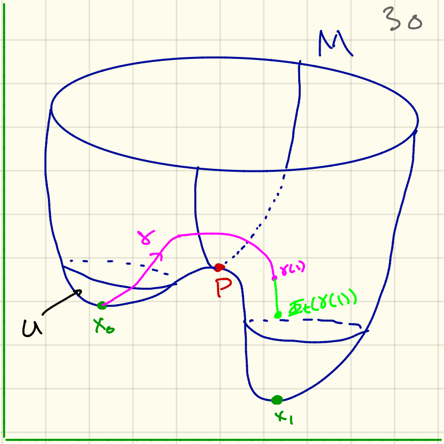
In all our examples, the collection $\mathcal{H}$ will be the same. Fix a gradient-like vector field X for f. (Note: We've only discussed this for Morse functions but it makes sense for any smooth f.) Denote by Φ_t the flow generated by $-X$. Condition (i) in the definition of Min-Max Data is clearly satisfied for the family $\mathcal{H}_f = \{\Phi_t : t \geq 0\}$.

Example: Take $\mathcal{S} = \{\{x\} : x \in M\}$. Then condition (ii) is satisfied and

$$c(\mathcal{H}_f, \mathcal{S}) = \min_{x \in M} f(x)$$

is an obvious critical value of f.

Example: Take $\mathcal{S} = \{M\}$. Then $c(\mathcal{H}_f, \mathcal{S}) = \max_{x \in M} f(x)$



The Mountain Pass Lemma

Suppose x_0 is a local minimum of f; that is, there is a small closed ball U centered at x_0 with $c_0 = f(x_0) < f(x)$, $x \in U \setminus \{x_0\}$. Note that $c_0' := \min_{x \in \partial U} f(x) > c_0$. Assume there is a point $x_1 \in M \setminus U$ with $c_1 = f(x_1) \le f(x_0)$. Denote by $\Gamma(x_0)$ the collection of all smooth paths $\gamma : [0,1] \to M$ such that $\gamma(0) = x_0$, $\gamma(1) \in M^{c_0} \setminus U$. The collection $\Gamma(x_0)$ is nonempty since M is connected and $x_1 \in M^{c_0} \setminus U$. Note that for any $\gamma \in \Gamma(x_0)$ and any $t \ge 0$, $\Phi_t \circ \gamma \in \Gamma(x_0)$. Now define $\mathcal{S} = \{ \gamma([0,1]) : \gamma \in \Gamma(x_0) \}$. Then $(\mathcal{H}_f, \mathcal{S})$ is a collection of min-max data for f. It follows that

$c = \inf_{\gamma \in \Gamma(x_0)} \sup_{s \in [0,1]} f(\gamma(s))$ is a critical value of f with $c \ge c_0' > c_0$. Critical points on the level set $f = c$ are called **mountain pass points**.

Note: The Mountain Pass Lemma implies that if a smooth function has two strict local minima then it must have a third critical point.

Index Theory

Fix a Riemannian metric on M. If $C \subset M$ is closed and $\varepsilon > 0$, denote by $N_\varepsilon(C)$ the open tube of radius ε around C. Denote the collection of closed subsets by $\mathcal{C}_M$.

Def: An **index theory** on M is a map

$$\gamma : \mathcal{C}_M \to \bar{\mathbb{Z}}_{\ge 0} = \{0, 1, 2, \dots\} \cup \{\infty\}$$

satisfying the following conditions.

- **Normalization** For every $x \in M$, there exists $r = r(x) > 0$ such that
$$\gamma(\{x\}) = 1 = \gamma(\overline{N_\varepsilon(x)}) \quad \forall x \in M, \ 0 < \varepsilon < r(x)$$
- **Topological Invariance** If $f : M \to M$ is a homeomorphism then
$$\gamma(C) = \gamma(f(C)) \quad \forall C \in \mathcal{C}_M$$
- **Monotonicity** If $C_0, C_1 \in \mathcal{C}_M$ and $C_0 \subseteq C_1$, then $\gamma(C_0) \le \gamma(C_1)$
- **Subadditivity** $\gamma(C_0 \cup C_1) \le \gamma(C_0) + \gamma(C_1)$

Given An Index Theory γ, Define

$$\Gamma_k = \{ C \in \mathcal{C}_M : \gamma(C) \geq k \}$$

The Axioms Of An Index Theory Imply That For Each k, The Pair $(\mathcal{H}_\delta, \Gamma_k)$ Is A Collection Of Min-Max Data. So For Each k The Min-Max Value

$$c_k = \inf_{C \in \Gamma_k} \max_{x \in C} f(x)$$

Is A Critical Value Of f. Since $\Gamma_1 \supset \Gamma_2 \supset \cdots$ It Follows That

$$c_1 \leq c_2 \leq \cdots$$

Note That The Family $\Gamma_1 \supset \Gamma_2 \supset \cdots$ Stabilizes At Γ_m, $m = \gamma(M)$. If It Happens That $c_1 < c_2 < \cdots < c_{\gamma(M)}$, Then We Can Conclude That f Has At Least $\gamma(M)$ Critical Points. This Need Not Be True, Though, But There Is A Remedy.

Prop: Suppose That For Some k, $p > 0$, We Have $c_k = c_{k+1} = \cdots = c_{k+p} = c$. Denote By K_c The Set Of Critical Points On The Level Set c. Then Either c Is An Isolated Critical Value Of f And K_c Contains At Least $p+1$ Points, Or c Is An Accumulation Point Of The Set Of Critical Values.

Proof: Assume c Is An Isolated Critical Value And Suppose K_c Contains At Most p Points. Then $\gamma(K_c) \leq p$. Set $T_r(K_c) = \overline{N_r(K_c)}$.

Deformation Lemma: Suppose c Is An Isolated Critical Value Of f And $K_c = \mathrm{Crit}(f) \cap \{f = c\}$ Is Finite. Then For Every $\delta > 0$, There Exist $\varepsilon > 0$, $r < \delta$, And A Homeomorphism $h = h_{\delta,\varepsilon,r}$ Of M Such That $h(\overline{M^{c+\varepsilon} \setminus T_r(K_c)}) \subset M^{c-\varepsilon}$.

Assuming This, Choose ε, r Sufficiently Small. Then The Normalization And Subadditivity Axioms Imply $\gamma(T_r(K_c)) \leq \gamma(K_c) \leq p$. Choose $C \in \Gamma_{k+p}$ So That

$$\max_{x \in C} f(x) \leq c_{k+p} + \varepsilon = c + \varepsilon.$$

Note That $C \subset T_r(K_c) \cup \overline{C \setminus T_r(K_c)}$ And From Subadditivity Of γ We See That

$$\gamma(\overline{C \setminus T_r(K_c)}) \geq \gamma(C) - \gamma(T_r(K_c)) \geq k. \quad \text{Thus,} \quad \gamma(h(\overline{C \setminus T_r(K_c)})) = \gamma(\overline{C \setminus T_r(K_c)}) \geq k$$

So That $C' = h(\overline{C \setminus T_r(K_c)}) \in \Gamma_k$. Since $\overline{C \setminus T_r(K_c)} \subset \overline{M^{c+\varepsilon} \setminus T_r(K_c)}$, The Deformation Lemma Implies That $C' \subset M^{c-\varepsilon}$. But $C' \in \Gamma_k \Rightarrow c = c_k \leq \max_{x \in C'} f(x)$, Which Is Impossible Since $C' \subset M^{c-\varepsilon}$. //

Proof Of Deformation Lemma: Technical, But It's Essentially The Same As Other Things We've Seen. Flow Along The Gradient. //

Cor: Suppose $\gamma: \mathcal{C}_M \to \mathbb{Z}_{\geq 0}$ Is An Index Theory On M. Then Any Smooth Function $f: M \to \mathbb{R}$ Has At Least $\gamma(M)$ Critical Points. //

How To Produce Index Theories?

Def: (a) A Subset $S \subset M$ Is **Contractible In M** If The Inclusion $S \hookrightarrow M$ Is Homotopic To A Constant Map.

(b) For A Closed $C \subset M$, Define Its **Lusternik-Schnirelmann Category** $\mathrm{cat}_M(C)$ To Be The Smallest Positive Integer k Such That There Is A Cover Of C By Closed Subsets $S_1, S_2, \ldots, S_k \subset M$ That Are Contractible In M. If Such A Cover Does Not Exist, Set $\mathrm{cat}_M(C) = \infty$.

Thm: If M Is Compact, Then The Correspondence $C \longmapsto \mathrm{cat}_M(C)$ Defines An Index Theory On M. (Also, $\mathrm{cat}(M) \geq \mathrm{CL}(M,R)+1$ Where $\mathrm{CL}(M,R)$ Denotes The Cuplength Of M With Coefficients In R.)

Proof: Normalization: $\mathrm{cat}_M(\{x\}) = 1 = \mathrm{cat}_M(\overline{N_\varepsilon(x)})$ For Sufficiently Small $\varepsilon > 0$.

Topological Invariance: If $C \in \mathcal{C}_M$ Is Covered By k Closed Sets $S_1, \ldots, S_k$, Then $h(C)$ Is Covered By $h(S_1), \ldots, h(S_k)$ And Each $h(S_i)$ Is Contractible In M.

Monotonicity: If $C_0 \subset C_1$ And C_1 Is Covered By $S_1, \ldots, S_k$, Then C_0 Is Also Covered By $S_1, \ldots, S_k$. It Follows That $\mathrm{cat}_M(C_0) \leq \mathrm{cat}_M(C_1)$.

Subadditivity: Cover C_0 By $S_1, \ldots, S_k$ And C_1 By $T_1, \ldots, T_\ell$. Then $C_0 \cup C_1$ Is Covered By The Union And So $\mathrm{cat}_M(C_0 \cup C_1) \leq k + \ell$. //

Cor: Any Smooth $f: M \to \mathbb{R}$ Has At Least $\mathrm{cat}(M)$ Critical Points, And Therefore At Least $\mathrm{CL}(M,R)+1$ Critical Points. //

Examples: $\mathrm{CL}(\mathbb{RP}^n, \mathbb{Z}/2) = \mathrm{CL}((S^1)^n, \mathbb{Z}) = \mathrm{CL}(\mathbb{CP}^n, \mathbb{Z}) = n$. It Follows That Any Smooth Map On These Has At Least $n+1$ Critical Points.

Cor: Every Even Map $f: S^n \to \mathbb{R}$ Has At Least $2(n+1)$ Critical Points.

Proof: f Descends To A Map $\bar{f}: \mathbb{RP}^n \to \mathbb{R}$ Which Has At Least $n+1$ Critical Points. Each Such Critical Point Is Covered By Exactly Two Critical Points Of f. //