Discrete Morse Theory

Morse Theory Is A Powerful Tool For The Study Of Smooth Manifolds. Can We Push These Ideas Into Larger Categories Of Spaces? Say, PL-Manifolds? Cell Complexes? Simplicial Complexes?

The PL Story Is Long And Complicated. One Might Try To Do Morse Theory In The Category Of Triangulated Manifolds But Technical Difficulties Arise Immediately. The Usual Type Of Function In This Context Is Piecewise-Linear, And These Are Far From Smooth. Some Things Can Be Said, But It's A Complicated Story.

In The Late 90's, Forman Introduced <u>Discrete Morse Theory</u> On Arbitrary Cell Complexes. We Will Restrict Our Attention To Regular Cell Complexes Here, But It Can Be Modified To Work On Arbitrary Complexes.

Def: Suppose X Is A CW-Complex. Suppose $\sigma^{(p)}$ Is A Face Of $\tau^{(p+1)}$ And Let $h: e^{(p+1)} \to X$ Be The Characteristic Map Of τ (i.e., h Maps $\mathrm{int}(e^{(p+1)})$ Homeomorphically Onto τ). We Say σ Is A Regular Face Of τ If

(i) $h: h^{-1}(\sigma) \to \sigma$ Is A Homeomorphism;

(ii) $\overline{h^{-1}(\sigma)}$ Is A Closed p-Ball.

A CW-Complex X Is Regular If All Its Faces Are Regular. Any Simplicial Complex Is A Regular CW-Complex. Note That If X Is Regular Then In The Cellular Chain Complex $C_*(X;\mathbb{Z})$ The Boundary Map $\partial: C_{p+1} \to C_p$ Has $\partial\tau = \sum a_\sigma \sigma$ With All $a_\sigma = 0, \pm 1$.

Prop: Let X Be A Regular CW Complex And Suppose That For Some p And $r \geq 1$ We Have $\tau^{(p+r)} > \nu^{(p-1)}$. Then There Are $(p+r-1)$-Cells σ And $\tilde{\sigma}$ Such That $\sigma \neq \tilde{\sigma}$ And $\tau > \sigma > \nu$, $\tau > \tilde{\sigma} > \nu$.

Proof: By Induction On r. Suppose $r=1$; i.e., We Have $\tau^{(p+1)} > \nu^{(p-1)}$. Since X Is Regular, The p-Cells In τ Are Dense In $\partial\tau$: $\overline{\bigcup_{\sigma^{(p)}<\tau} \sigma} = \bar{\tau} - \tau$. Thus There Is A p-Cell σ With $\tau > \sigma > \nu$. Now Choose An Orientation On Each Cell In X And Write

$\partial\tau = \pm\sigma + \sum_{\substack{\tilde{\sigma}\neq\sigma \\ \tilde{\sigma}<\tau}} c_{\tilde{\sigma}}\tilde{\sigma}$. Since $\nu < \sigma$ We May Also Write $\partial\sigma = \pm\nu + \sum_{\tilde{\nu}\neq\nu} c_{\tilde{\nu}}\tilde{\nu}$.

Then $0 = \partial^2\tau = \pm\partial\sigma + \sum_{\tau>\tilde{\sigma}\neq\sigma} c_{\tilde{\sigma}}\partial\tilde{\sigma} = \pm\nu + \sum_{\tau>\tilde{\sigma}\neq\sigma} c_{\tilde{\sigma}}\partial\tilde{\sigma} + \sum_{\tilde{\nu}\neq\nu} c_{\tilde{\nu}}\tilde{\nu}$.

For This To Hold, There Must Be Some $\tilde{\sigma}$, With $\tau > \tilde{\sigma} \neq \sigma$ Satisfying

$\partial\tilde{\sigma} = c\nu + (\text{Sum of } (p-1)\text{-Cells Other Than } \nu)$ For Some $c \neq 0$. This Implies $\tilde{\sigma} > \nu$.

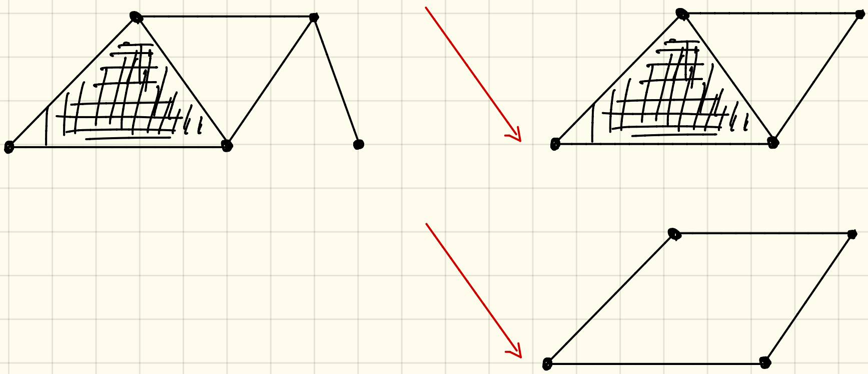
For $r>1$, we again have that the $(p+r-1)$-cells in $\partial\tau$ are dense. So we can find a $(p+r-1)$-cell σ with $\tau > \sigma > \upsilon^{(p-1)}$. Similarly, we can find a $(p+r-2)$-cell $\tilde{\sigma}$ with $\sigma > \tilde{\sigma} > \upsilon$. Applying the base case to the triple $\tau > \sigma > \tilde{\sigma}$ we find a $(p+r-1)$-cell $\bar{\sigma} \neq \sigma$ with $\tau > \bar{\sigma} > \tilde{\sigma}$. The cells σ & $\bar{\sigma}$ have the required properties.

Def: Suppose X is a CW-complex and $\sigma^{(p)} < \tau^{(p+1)}$ are cells such that

(a) σ is a regular face of τ;

(b) σ is not a face of any other cell.

Let $Y = X - (\sigma \cup \tau)$. We say **$X$ collapses onto Y**. More generally we say M collapses onto N and write $M \searrow N$ if M can be transformed into N by a finite sequence of such operations.



Note that if $M \searrow N$, then N is a deformation retract of N. If $M \searrow \{v\}$ for some vertex v we say that M is **collapsible**. Note that collapsible complexes are contractible, but not conversely.

Def: Let X be a finite (regular) CW complex. A **discrete Morse function** on X is a function $f : X \to \mathbb{R}$ satisfying, for all $\sigma^{(p)}$,

(i) $\#\{\tau^{(p+1)} > \sigma \mid f(\tau) \leq f(\sigma)\} \leq 1$

(ii) $\#\{\upsilon^{(p-1)} < \sigma \mid f(\upsilon) \geq f(\sigma)\} \leq 1$.

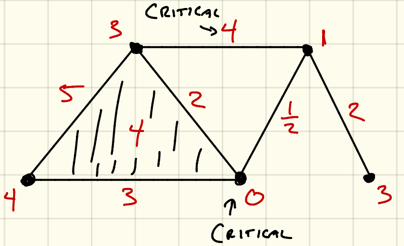
Note that we are abusing notation here. The function f is defined on the set of cells of X, not X itself.

Def: A cell $\sigma^{(p)}$ is a **critical cell of index p** if

(i) $\#\{\tau^{(p+1)} > \sigma \mid f(\tau) \leq f(\sigma)\} = 0$

(ii) $\#\{\upsilon^{(p-1)} < \sigma \mid f(\upsilon) \geq f(\sigma)\} = 0$.

Example:



Note: The minimum of f must occur at a vertex, which then must be a critical cell of index 0. Indeed, if $p \geq 1$ then every p-cell has at least 2 $(p-1)$-faces. At most one can have a larger value, so the global minimum can occur only at a vertex.

Similarly, if X is a triangulated n-manifold, then the maximum of f must occur on an n-simplex, which is then a critical cell of index n. Again this follows from the fact that for each $p \leq n-1$, every p-cell is a face of at least 2 $(p+1)$-cells.

Def: A cell σ is regular if it is not critical.

Note that σ is regular if and only if one of the following holds:

(i) $\exists\, \tau^{(p+1)} > \sigma$ such that $f(\tau) \leq f(\sigma)$

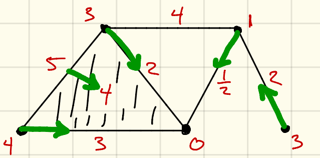
(ii) $\exists\, \nu^{(p-1)} < \sigma$ such that $f(\nu) \geq f(\sigma)$.

Lemma: Conditions (i) and (ii) cannot both be true.

Proof: Condition (ii) requires $p \geq 1$. Suppose (i) is true. If $\tilde{\sigma} \neq \sigma$ is any other p-face of τ then we must have $f(\tilde{\sigma}) < f(\tau)$. In particular, $f(\tilde{\sigma}) < f(\sigma)$. Now, suppose (ii) is also true: there is a $\nu^{(p-1)} < \sigma$ with $f(\nu) \geq f(\sigma)$. We know there is a $\tilde{\sigma} \neq \sigma$ with $\tau > \tilde{\sigma} > \nu$. By definition, $f(\nu)$ cannot be $\geq$ both $f(\sigma)$ and $f(\tilde{\sigma})$. Thus, $f(\nu) < f(\tilde{\sigma})$. But then $f(\sigma) \leq f(\nu) < f(\tilde{\sigma}) < f(\tau) \leq f(\sigma)$, a contradiction. //

The Discrete Gradient Vector Field

Note that regular faces occur in pairs: σ regular $\Rightarrow \exists\, \tau > \sigma$ with $f(\tau) \leq f(\sigma)$ or $\exists\, \nu < \sigma$ with $f(\nu) \geq f(\sigma)$. Denote by V the collection of all such pairs $(\alpha^{(p)} < \beta^{(p+1)})$. We call V the discrete gradient vector field associated to f.



Gradient vectors shown in green. Arrow points from lower-dimensional face to coface with lower function value.

Def: A Discrete Vector Field is a map $W: K \to K \cup \{0\}$ (K = Set of Cells of X) satisfying

(i) For each p, $W(K_p) \subseteq K_{p+1} \cup \{0\}$ (K_p = Set of p-Cells)

(ii) For each $\sigma^{(p)} \in K_p$, either $W(\sigma) = 0$ or σ is a regular face of $W(\sigma)$

(iii) If $\sigma \in \mathrm{Im}(W)$, then $W(\sigma) = 0$

(iv) For each $\sigma^{(p)} \in K_p$, $\#\{\nu^{(p-1)} \in K_{p-1} \mid W(\nu) = \sigma\} \leq 1$.

Def: A W-Path of Dimension p is a sequence

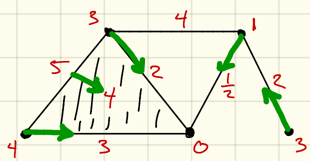
$$\gamma = \sigma_0^{(p)} < \tau_0^{(p+1)} > \sigma_1^{(p)} < \tau_1^{(p+1)} > \cdots < \tau_{r-1}^{(p+1)} > \sigma_r^{(p)}$$

with $W(\sigma_i) = \tau_i$ and $\sigma_i \neq \sigma_{i+1}$, $i = 0, \ldots, r-1$. Call γ a Closed Path if $\sigma_0 = \sigma_r$ and Non-Stationary if $\sigma_1 \neq \sigma_0$.

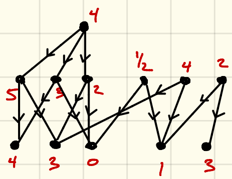
Example: If $f: X \to \mathbb{R}$ is a discrete Morse function, then its gradient is a discrete vector field. We denote this by V_f.

Thm: Let W be a discrete vector field. Then there is a discrete Morse function f with $W = V_f \iff W$ has no non-stationary closed paths. Moreover, for every such W, f can be chosen to have the property that if $\sigma^{(p)}$ is critical, then $f(\sigma) = p$ (i.e. f is Self-Indexing).

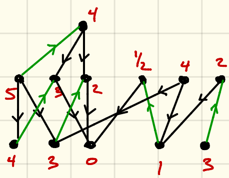
Proof: We follow the approach of Chari. Recall the Hasse Diagram of the complex X: this is the directed graph with vertex set K = Set of Cells of X and edges $\tau^{(p+1)} \to \sigma^{(p)}$ when σ is a codimension-1 face of τ.



X



Hasse Diagram



Modified Diagram

Given W, modify the Hasse diagram of X as follows: If $W(\sigma^{(p)}) = \tau^{(p+1)}$, reverse the arrow $\tau^{(p+1)} \to \sigma^{(p)}$. Call this new directed graph H.

Lemma: Suppose $W = V_f$ for some discrete Morse function f. Then a sequence of cells $\alpha_0 < \beta_0 > \alpha_1 < \beta_1 > \cdots < \beta_{r-1} > \alpha_r$ is a W-path $\iff f(\alpha_0) \geq f(\beta_0) > f(\alpha_1) \geq f(\beta_1) > \cdots \geq f(\beta_{r-1}) > f(\alpha_r)$

Proof: Suppose the given sequence is a W-path. Then by definition $f(\alpha_i) \geq f(\beta_i)$ for all i. Moreover, since α_{i+1} is not paired with β_i, $f(\beta_i) > f(\alpha_{i+1})$. Conversely, if a sequence of simplices satisfies the chain of inequalities, then $(\alpha_i, \beta_i) \in W$ by definition. Moreover, since β_i is the unique such simplex, each α_i, β_i occur in just one paired W-path. //

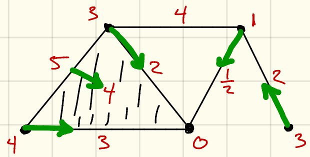
Now, Suppose $W = V_f$ For Some f. If We Had A Closed W-Path

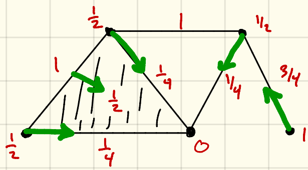
$$\alpha_0 < \beta_0 > \alpha_1 < \beta_1 > \cdots < \beta_{r-1} > \alpha_r = \alpha_0$$

Then The Lemma Implies $f(\alpha_0) \geq f(\beta_0) > f(\alpha_1) \geq f(\beta_1) > \cdots \geq f(\beta_{r-1}) > f(\alpha_r) = f(\alpha_0)$,

Which Is Impossible.

Conversely, Suppose W Has No Closed Paths. This Implies That The Modified Hasse Diagram H Has No Directed Cycles. We Now Employ A Well-Known Result From Graph Theory: A Directed Graph G Is Acyclic $\Leftrightarrow$ There Is A Function $f: \mathrm{Vert}(G) \to \mathbb{R}$ Which Is Strictly Decreasing Along Each Directed Path. Such A Function On H Is A Discrete Morse Function On X. Moreover, It Is Clear That We Can Set $f(\sigma^{(p)}) = p$ For Any Critical Cell σ. //





Self-Indexing Function

The Main Theorems

Def: For A Real Number c Define $X^c = \bigcup_{\substack{\sigma \in K \\ f(\sigma) \leq c}} \bigcup_{\tau \leq \sigma} \tau$. This Is The Level Subcomplex.

Lemma: Let σ Be A p-Cell And Suppose $\tau > \sigma$. Then There Is A $(p+1)$-Cell $\tilde{\tau}$ With $\sigma < \tilde{\tau} \leq \tau$ And $f(\tilde{\tau}) \leq f(\tau)$.

Proof: Since $\tau > \sigma$, $\dim \tau > \dim \sigma$. If $\dim \tau = p+1$, We Can Take $\tilde{\tau} = \tau$. Assume $\dim \tau = p+r$, $r > 1$. Then There Are Two $(p+r-1)$-Faces ν_1, ν_2 Satisfying $\tau > \nu_1 > \sigma$, $\tau > \nu_2 > \sigma$. By Definition, Either $f(\nu_1) < f(\tau)$ Or $f(\nu_2) < \tau$. In Either Case The Result Follows By Induction. //

Thm: If $a < b$ Are Real Numbers Such That $[a, b]$ Contains No Critical Values Of f, Then $X^b \searrow X^a$.

Proof: Note That If $\tau^{(p+1)} > \sigma^{(p)}$ Satisfies $f(\tau) \leq f(\sigma)$ Then We May Perturb f By Replacing $f(\tau)$ By $f(\tau) - \varepsilon$ Or $f(\sigma)$ By $f(\sigma) + \varepsilon$, $\varepsilon \geq 0$ Small, Without Changing Which Cells Are Critical. Doing This Repeatedly We May Perturb f Slightly Without Changing X^b Or X^a So That $f: X \to \mathbb{R}$ Is 1-1.

If $f^{-1}([a,b]) = \emptyset$ Then $X^b = X^a$ And There Is Nothing To Prove. Otherwise, By Partitioning $[a,b]$ If Necessary, We May Assume There Is A Single Noncritical Cell σ With $f(\sigma) \in [a,b]$. Exactly One Of The Following Holds:

(i) $\exists\, \tau^{(p+1)} > \sigma$ With $f(\tau) \leq f(\sigma)$

(ii) $\exists\, \nu^{(p-1)} < \sigma$ With $f(\nu) \geq f(\sigma)$.

In Case (i), $f(\tau) \leq f(\sigma) \Rightarrow \tau \in X^a$. Since $\sigma < \tau$, $\sigma \in X^a$ As Well. Thus, $X^a = X^b$ And There Is Nothing To Prove.

In Case (ii), We Know Case (i) Is Not True. So For All $\tau^{(p+1)} > \sigma$ We Have $f(\tau) > f(\sigma)$. In Particular $f(\tau) > b$. It Follows That For Any $\tau > \sigma$, $f(\tau) > b$ And So $\sigma \cap X^a = \emptyset$. We've Assumed There Is A $\nu^{(p-1)} < \sigma$ With $f(\nu) > f(\sigma)$ So That $f(\nu) > b$. If $\tilde{\nu}^{(p-1)}$ Is Any Other $(p-1)$-Face Of σ We Must Have $f(\tilde{\nu}) < f(\sigma)$ So That $f(\tilde{\nu}) < a$. Thus, $\tilde{\nu}$ And All Its Faces Are In X^a. Let $\tilde{\sigma}^{(p)}$ Be Any Other p-Cell With $\tilde{\sigma} > \nu$. Then $f(\tilde{\sigma}) > f(\nu) > b$ And For Any $\tilde{\sigma}$ Of Any Dimension With $\tilde{\sigma} > \nu$ $f(\tilde{\sigma}) > b \Rightarrow \nu \cap X^a = \emptyset$. Thus, $X^b = X^a \cup \sigma \cup \nu$ And ν Is A Free Face Of σ. Thus $X^b \searrow X^a$. //

Thm: Suppose $\sigma^{(p)}$ Is A Critical Cell Of Index p With $f(\sigma) \in [a,b]$ And $f^{-1}([a,b])$ Contains No Other Critical Cells. Then $X^b \simeq X^a \cup_{\partial e^p} e^p$.

Proof: Again We May Assume f Is 1-1 And So We Can Find $a < a' < b' < b$ With $\sigma = f^{-1}([a',b'])$. We Know $X^b \searrow X^{b'}$ And $X^{a'} \searrow X^a$ So It Suffices To Show $X^{b'} = X^{a'} \cup_{\partial e^p} e^p$. Since σ Is Critical, If $\tau^{(p+1)} > \sigma$ We Have $f(\tau) > f(\sigma)$ So That $f(\tau) > b' \Rightarrow$ For Any Cell τ With $\tau > \sigma$ We Have $f(\tau) > b'$. Thus, $\sigma \cap X^{a'} = \emptyset$. We Also Know That For Every $\nu^{(p-1)} < \sigma$ We Have $f(\nu) < f(\sigma)$ + So $f(\nu) < a'$. Thus $\nu \subseteq X^{a'}$ And So $\partial\sigma \subseteq X^{a'}$. It Follows That $X^{b'} = X^{a'} \cup_{\partial\sigma} \sigma$, And Since σ Is Homeomorphic To e^p The Result Follows. //

Note: This Is Much Simpler Than The Proof Of The Analogous Result In Smooth Morse Theory.

Cor: For Each p, Let c_p Denote The Number Of Critical Points Of Index p For f. Then X Is Homotopy Equivalent To A CW-Complex With c_p Cells Of Dimension p. //

Cor: (Strong Morse Inequalities) For Any $k \geq 0$: $c_k - c_{k-1} + \cdots \pm c_0 \geq \beta_k - \beta_{k-1} + \cdots \pm \beta_0$.

Cor: (Weak Morse Inequalities) (a) $c_k \geq \beta_k$ For All k; (b) $\chi(X) = \sum_{j=0}^{n} (-1)^j c_j$.

Examples

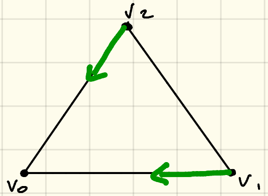
1. **The n-Simplex** This space is contractible, but even better, it is <u>Collapsible</u>. In fact:

Prop: Let X be a regular cell complex. Then the cone CX is collapsible.

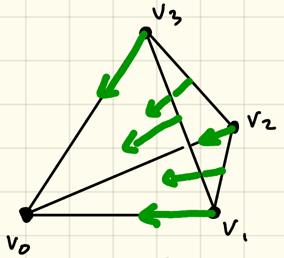
Proof: Recall that $CX = X * \{w\}$ where w is a point. The cells of CX are precisely the various $\sigma * w$ for σ a cell in X along with w. Define a vector field V on CX by $V(\sigma) = \sigma * w$. This has exactly one critical cell, namely the vertex w. V is acyclic: Suppose $\alpha_0 < \beta_0 > \alpha_1 < \beta_1 > \cdots < \beta_{r-1} > \alpha_r = \alpha_0$ is a closed V-path. Then for each i, $\beta_i = \alpha_i * w$ and $\alpha_{i+1} = \nu_{i+1} * w$ for some face $\nu_{i+1} < \alpha_i$. In particular, $\alpha_0 = \nu * w$ for some face $\nu < \alpha_0$, which is impossible. Now, to collapse CX to w, note that a top dimensional cell has the form $\tau * w$ where τ is a top dimensional cell of X; τ is a free face of this cell. Remove the cells in decreasing order of dimension. This gives a collapse $CX \searrow w$.

In particular, since $\Delta^n = C\Delta^{n-1}$, and $\Delta^0 = *$, we see that Δ^n is collapsible.

2. **The n-Sphere** Let's consider $S^n = \partial \Delta^{n+1}$. Denote the vertices of Δ^{n+1} by $v_0, \dots, v_{n+1}$. Define a vector field V by $V(v_0) = \emptyset$ and $V(\langle v_{i_1}, \dots, v_{i_k}\rangle) = \langle v_0, v_{i_1}, \dots, v_{i_k}\rangle$ for simplices $\langle v_{i_1}, \dots, v_{i_k}\rangle$ with $i_1 \neq 0$ (always write indices in increasing order). Note that the n-simplex $\langle v_1, \dots, v_{n+1}\rangle$ is critical for V, but every other simplex is paired with exactly one coface. V is acyclic: Paths can't get started: If $\alpha_0 < \beta_0 > \alpha_1$, then either $\alpha_1 = \alpha_0$ or $\alpha_1 = \langle v_0, v_{j_1}, \dots, v_{j_{k-1}}\rangle$ and so it is paired with a lower-dimensional face. Thus, S^n has the homotopy type of a CW-complex of the form $e^0 \cup e^n$.



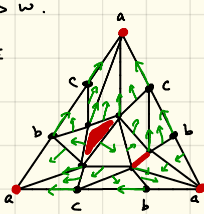
$n = 1$



$n = 2$

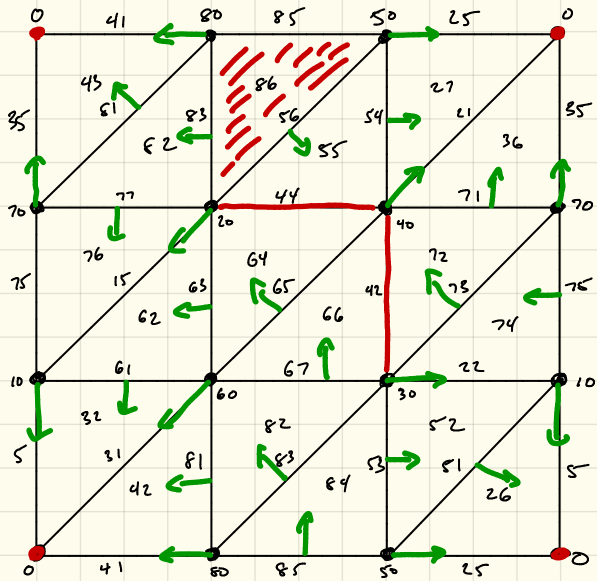
Another way to think of this is to view S^n as $CS^{n-1} \cup e^n$ where $S^{n-1} = \partial\langle v_1, \dots, v_{n+1}\rangle \subset S^n$. The vector field V is essentially the same as the one we used to show $CX \searrow w$.

3. **The Dunce Cap** This space is contractible but not collapsible. This is because no 2-simplex has a free face. That means that any discrete gradient on X must have a critical 2-simplex. There also must be at least one critical vertex. So, $c_2 \geq 1$ and $c_0 \geq 1$. Suppose $c_0 = c_2 = 1$. Then the Morse inequalities force $1 = \chi(X) = c_2 - c_1 + c_0 = 2 - c_1 \Rightarrow c_1 = 1$. Here's an example.



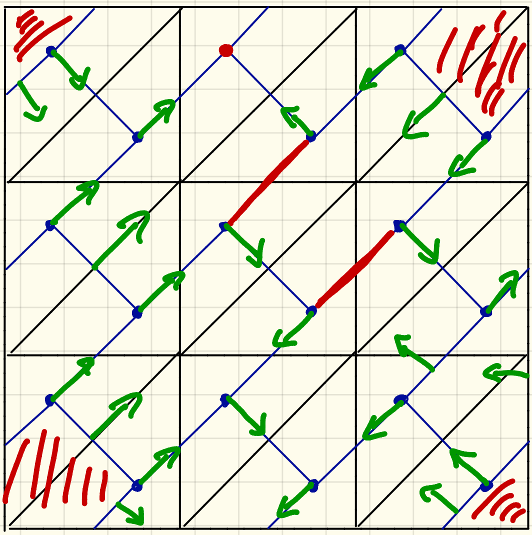
4. The Torus

Here's A Discrete Morse Function On The Torus:



5. The Dual Vector Field

Suppose X Is A Triangulation Of A Closed Manifold. Consider The Dual Cell Complex X^*: Its Vertices Correspond To The n-Cells In X. Two Vertices Are Joined By An Edge If The Corresponding n-Cells Share A Face, etc. If σ Is An i-Simplex Then The Dual Cell σ^* Is An $(n-i)$-Cell. Example:



The Dual Cell Decomposition Of The Torus Is Shown In Blue. The Dual Vector Field $-V$ Is Defined By $\{\alpha < \beta\} \in V \Leftrightarrow \{\beta^* < \alpha^*\} \in -V$

Note That σ Is Critical For $V \Leftrightarrow \sigma^*$ Critical For $-V$. It Follows Then That If C_p Denotes The # Of Critical p-Cells Then

$$C_p(V) = C_{n-p}(-V)$$

This Suggests A Path For Proving Poincaré Duality (Later).

Lemma: Let Y be a subcomplex of X. If $f: X \to \mathbb{R}$ is a discrete Morse function then the restriction $f|_Y : Y \to \mathbb{R}$ is a discrete Morse function on Y. Moreover, if $\sigma \in Y$ is critical for f then σ is critical for $f|_Y$.

Proof: Immediate from definitions.//

Lemma: Suppose Y is a subcomplex of X and $f: Y \to \mathbb{R}$ is a discrete Morse function. Then f extends to $\tilde{f}: X \to \mathbb{R}$.

Proof: Let $c = \max_{\sigma \in Y} f(\sigma)$. If τ is a cell in $X - Y$, set $\tilde{f}(\tau) = c + \dim \tau$, and if $\sigma \in Y$ set $\tilde{f}(\sigma) = f(\sigma)$. Then $\tilde{f}$ is a discrete Morse function on X.//

Lemma: Suppose Y is a subcomplex of X and $X \searrow Y$. Let f be a discrete Morse function on Y and let $c = \max_{\sigma \in Y} f(\sigma)$. Then f extends to $\tilde{f}: X \to \mathbb{R}$ such that $Y = X^c$ and $\tilde{f}$ has no critical cells in $X - Y$.

Proof: Suppose $X = Y \cup \sigma \cup \tau$ where τ is a free face of σ. Define $\tilde{f} = f$ on Y and set $\tilde{f}(\sigma) = c+1$, $\tilde{f}(\tau) = c+2$. The general result follows by induction on the number of collapses required.//

Note that this implies that Δ^n supports a discrete Morse function with exactly one critical point ($\Delta^n \searrow v$) and $\partial\Delta^n$ supports a function with exactly two critical cells (choose any $(n-1)$-simplex σ and note that $\partial\Delta^n - \sigma \searrow v$).

Thm: Suppose X is a regular cell complex and let f be a discrete Morse function on X with exactly two critical cells. Then X is homotopy equivalent to a sphere.

Proof: Since X is connected, the weak Morse inequalities imply that at least one critical cell is a vertex. If the other critical cell has dimension n, then $X \simeq e^0 \cup e^n = S^n$.//

Note: If X is a compact PL-manifold, then X is in fact piecewise linear equivalent to $\partial\Delta^{n+1}$. Indeed, since $H_n(X; \mathbb{Z}/2) \neq 0$, f must have a critical n-cell σ (unique). Let $Y = X - \sigma$. Then $f|_Y$ has a single critical vertex v and $Y \searrow v$. By Whitehead's Thm, Y is a PL n-cell and since $X = Y \cup_\partial \bar{\sigma}$ and $\bar{\sigma}$ is a PL-n-cell, it follows that X is a PL-n-sphere.//

Also: Assuming X is topologically a manifold without boundary, X is <u>homeomorphic</u> to S^n by the Poincaré Conjecture.

Thm: Suppose X is a triangulated connected closed n-manifold. Then X has a DMF with a single critical vertex and a single critical n-simplex.

Proof: Note that the 1-skeleton $X^{(1)}$ is a connected graph. Let T be a maximal tree. If v is any vertex then $T \searrow v$ and so T has a DMF with a single critical vertex, v. Extend this to $f: X \to \mathbb{R}$. Since $X - T$ has no vertices, v is the only critical vertex for f. If $\dim X = 0$, we are done. If $\dim X = 1$, then X is a circle. If e is any edge, then $X - e \searrow v$. Set $c = \max_{\sigma \in X - e} f(\sigma)$ & set $f(e) = c+1$. If $\dim X \geq 2$, let σ be any n-simplex and let $Y = X - \sigma$. Y is an n-mfld with boundary and $Y \searrow Z$ where Z is a subcomplex of $\dim \leq n-1$. Z supports a DMF with a single crit. vertex v. Then f extends to Y with no further crit cells. Set $f(\sigma) = \max f|_Y + 1$.//

Canceling Critical Cells

Recall The Smooth Case: If p + q Are Critical Points Of $f: M \to \mathbb{R}$ With $\operatorname{ind}(q) = \operatorname{ind}(p)+1$ And The Ascending/Descending Manifolds Intersect Transversely, Then The Critical Points Can Be "Canceled". The Proof Of This Fact Is Very Technical And Long.

In The Discrete Case, It Makes No Sense To Talk About "Perturbations" Since Small Variations Generally Change Nothing. Instead We Focus On The Gradient.

Thm: Suppose V Is A Discrete Gradient On X. Suppose $\tau^{(p+1)}$ And $\sigma^{(p)}$ Are Critical Cells Such That There Is A Face $\nu^{(p)} < \tau$ And A Unique V-Path

$$\tau > \nu = \sigma_0 < \tau_0 > \sigma_1 < \tau_1 > \cdots < \tau_r > \sigma_r = \sigma.$$

Then There Is A Gradient Vector Field W Such That The Set Of Critical Cells Of W Is The Set Of Critical Cells Of V With σ + τ Removed. Moreover, $W = V$ Except Along The Unique Path Above.

Proof: This Is Almost Trivial. Simply Turn V Around Along The Path. Define W As Follows:

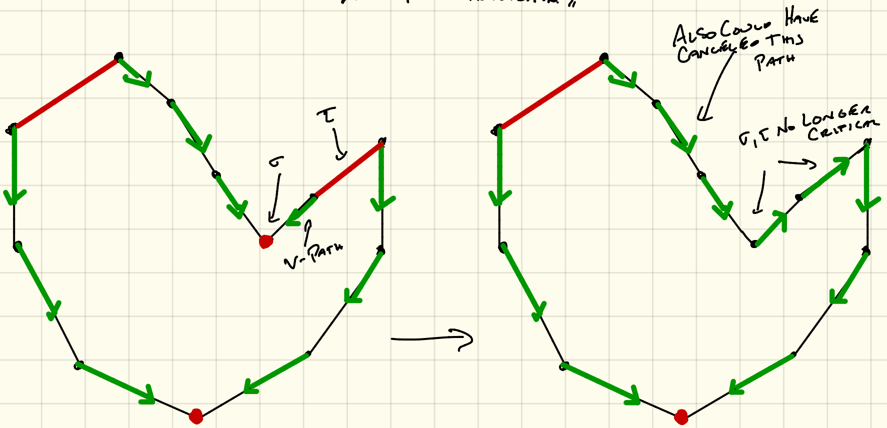
$$W(\alpha) = V(\alpha) \text{ If } \alpha \notin \{\nu, \tau_0, \sigma_1, \tau_1, \dots, \tau_r, \sigma\}$$
$$W(\sigma_i) = \tau_{i-1}, \quad i = 1, \dots, r$$
$$W(\nu) = \tau.$$

We Need Only Show There Are No Nontrivial Closed W-Paths. Any Such Path Would Have To Contain A p-Cell In The Path Above And A p-Cell Not In The Path. Thus A Closed Path Would Contain A Segment Of The Form $\sigma_i < \eta_0 > \mu_1 < \eta_1 > \cdots < \eta_s > \sigma_j$ With $s \geq 0$ And $\eta_i, \mu_i \neq \sigma_i, \tau_i$ For All i.

Since $W(\eta_i) = V(\eta_i)$ And $W(\mu_i) = V(\mu_i)$ For All i, We Then Have A V-Path $\eta_0 > \mu_1 < \eta_1 > \cdots < \eta_s > \sigma_j$

If $i \neq 0$, Then $\mu_1 \neq \sigma_{i-1}, \sigma_i$ And $\mu_1 < W(\sigma_i) = \tau_{i-1}$. We May Then Insert The Segment Into The Original Path:

$\nu = \sigma_0 < \tau_0 > \sigma_1 < \cdots > \sigma_{i-1} < \tau_{i-1} > \mu_1 < \eta_1 > \cdots > \sigma_j < \tau_j > \cdots > \sigma_r = \sigma$, A Second V-Path From $\partial\tau$ To σ. This Is A Contradiction. If $i = 0$, Then $\nu \neq \mu_1 < W(\nu) = \tau$ And We May Replace The Initial Segment Of The Path With The Segment To Obtain A Second V-Path From $\partial\sigma$ To τ, Also A Contradiction. //



Homology

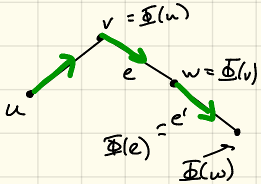
Since A (Discrete) Morse Function Gives Us A Cell Decomposition Of The Space, It Is Natural To Ask About The Boundary Maps In The Cellular Chain Complex. This Is A Rather Subtle Question. In The Smooth Case, One First Requires The Function To Be Morse-Smale, A Technical Condition About Transverse Intersections Of Descending/Ascending Discs. This Yields A Cell Decomposition Of The Manifold (The Morse-Smale Decomposition) And Then One Considers Equivalence Classes Of Gradient Paths From Index p Critical Points To Index $p-1$ Critical Points.

In The Discrete Case, Something Similar Happens, But It Is Easier To Describe. Suppose X Is A Regular Cell Complex, V Is A Discrete Gradient On X, And f Is A Compatible Discrete Morse Function. Choose An Orientation On Each Cell In X And Write, For A p-Cell σ, $\partial\sigma = \sum_{\alpha^{(p-1)}<\sigma} \varepsilon(\sigma,\alpha)\alpha$, Where $\varepsilon(\sigma,\alpha)$ Is The Incidence Number Of α In The Boundary Of σ, Counted With Multiplicity. (Note: Since X Is Regular, $\varepsilon(\sigma,\alpha)=0,\pm1$ For All Pairs $\alpha<\sigma$.) Define An Inner Product $\langle -,- \rangle$ On The Chains $C_\bullet(X;\mathbb{Z})$ By Declaring All Cells To Be Orthonormal. Then We May Write $\partial\sigma = \sum_{\alpha<\sigma} \langle \partial\sigma,\alpha\rangle\alpha$.

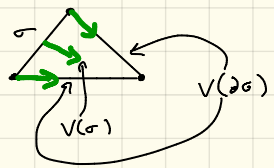
Define $V: C_p(X;\mathbb{Z}) \to C_{p+1}(X;\mathbb{Z})$ As Follows. If $\{\sigma<\tau\}\in V$, Set $V(\sigma) = -\langle \partial\tau,\sigma\rangle\tau$. If There Is No Such τ (e.g. σ Critical) Set $V(\sigma)=0$. Extend This Linearly To $C_p \to C_{p+1}$.

The Discrete Flow Φ

Think About The Flow Lines Associated To $-\nabla g$ For A Morse Function $g: M\to\mathbb{R}$. The Critical Points Of g Are Fixed By The Flow, But Non Critical Points Flow Down. How Does This Discretize? First Consider Vertices: If $v\in X$ Is Critical, Then $\Phi(v)=v$ (i.e. v Is Fixed). If v Is Not Critical, And $V(v)=\pm e$ Then v Should Flow To The Other Vertex Of e: $\Phi(v) = v + \partial(V(v))$.



In General, For A p-Cell σ, Set $\Phi(\sigma) = \sigma + \partial V(\sigma) + V(\partial\sigma)$. This Makes Sense: $V(\sigma)$ Is The Component Of $-\nabla f$ Transverse To σ And The Component Tangent To σ Is Determined By $V(\partial\sigma)$:



Φ Extends Linearly To A Map $\Phi: C_p(X;\mathbb{Z}) \to C_p(X;\mathbb{Z})$

Note The Following Properties Of V:

(i) If σ Is An Oriented p-Cell, $\#\{\alpha^{(p-1)}<\sigma \mid V(\alpha)=\pm\sigma\}\le 1$

(ii) σ Is Critical $\iff$ $\sigma\notin \operatorname{im}(V)$ And $V(\sigma)=0$

Lemma: $V\circ V = 0$

Proof: If $V(\alpha)=\pm\sigma$, Then There Is No Cell $\tau>\sigma$ With $V(\sigma)=\tau \Rightarrow V(\sigma)=0$. //

Prop. (i) $\Phi\partial = \partial\Phi$

(ii) If $\sigma_1,\dots,\sigma_r$ Are The Oriented p-Cells Of X And We Write $\Phi(\sigma_i) = \sum a_{ij}\sigma_j$ Then

(a) For Each i, $a_{ii}=0$ or 1 And $a_{ii}=1 \iff \sigma_i$ Is Critical

(b) If $i\ne j$ And $a_{ij}\ne 0$, Then $f(\sigma_j)<f(\sigma_i)$

Proof: Since $\Phi = 1 + \partial V + V\partial$, we have
$$\Phi\partial = (1+\partial V + V\partial)\partial = \partial + \partial V \partial + V\partial^2 = \partial + \partial V \partial$$
$$\partial\Phi = \partial(1+\partial V + V\partial) = \partial + \partial^2 V + \partial V \partial = \partial + \partial V\partial$$
If σ is a p-cell, then σ satisfies either (i) σ is critical; (ii) $\pm\sigma \in \mathrm{im}(V)$; or (iii) $V(\sigma)\neq 0$ (and these are exclusive). If σ is critical, then $V(\sigma)=0$ and $\Phi(\sigma) = \sigma + V(\partial\sigma) = \sigma + \sum_{\alpha<\sigma} \langle \partial\sigma, \alpha\rangle V(\alpha)$. Since σ is critical, each such α satisfies $f(\alpha) < f(\sigma)$. Moreover, for each such α, $V(\alpha) = 0$ or $V(\alpha) = \beta^{(p)}$ with $f(\beta) \le f(\alpha) < f(\sigma)$. We therefore have $\Phi(\sigma) = \sigma + \sum a_\beta \beta$ where $a_\beta \neq 0 \Rightarrow f(\beta) < f(\sigma)$.

If $\pm\sigma \in \mathrm{im}(V) \subseteq \ker(V)$, then $\Phi(\sigma) = \sigma + V(\partial\sigma) = \sigma + \sum_{\alpha<\sigma} \langle \partial\sigma, \alpha\rangle V(\alpha)$. There is exactly one α with $V(\alpha) = \pm\sigma$ and $\langle \partial\sigma, \alpha\rangle V(\alpha) = -\sigma$. Also, if $\tilde\alpha$ is another face of σ, we have $V(\tilde\alpha) = 0$ or $V(\tilde\alpha) = \tilde\sigma$ with $f(\tilde\sigma) \le f(\tilde\alpha) < f(\sigma)$. It follows that $\Phi(\sigma) = \sum_{\tilde\sigma^{(p)}} a_{\tilde\sigma}\tilde\sigma$ where $a_{\tilde\sigma} \neq 0 \Rightarrow f(\tilde\sigma) < f(\sigma)$.

Finally, if $V(\sigma) = -\langle \partial\tau, \sigma\rangle\tau \neq 0$, then $\Phi(\sigma) = \sigma + V(\partial\sigma) + \partial(V(\sigma))$. Since $V(\sigma)\neq 0$, $\pm\sigma \notin \mathrm{im}(V)$, and so for each $(p-1)$-face $\alpha < \sigma$, either $V(\alpha) = 0$ or $V(\alpha) = \pm\tilde\sigma$, where $f(\tilde\sigma) \le f(\alpha) < f(\sigma)$. Also,
$\partial(V(\sigma)) = -\langle\partial\tau,\sigma\rangle\partial\tau = -\langle\partial\tau,\sigma\rangle^2\sigma + \sum a_{\tilde\sigma}\tilde\sigma = -\sigma + \sum b_{\tilde\sigma}\tilde\sigma$ where $b_{\tilde\sigma}\neq 0$ implies $f(\tilde\sigma) \le f(\alpha) < f(\sigma)$.//

Now, denote by $K_p(X;\mathbb{Z})$ the Φ-invariant chains:
$$K_p(X;\mathbb{Z}) = \{c \in C_p(X;\mathbb{Z}) : \Phi(c) = c\}.$$
The boundary map restricts to a boundary map on $K_\bullet(X;\mathbb{Z})$. I claim the homology of $K_\bullet(X;\mathbb{Z})$ is $H_\bullet(X;\mathbb{Z})$. We have the inclusion $i: K_p(X;\mathbb{Z}) \to C_p(X;\mathbb{Z})$. We need a map in the other direction.

Lemma: Let $c \in K_p(X;\mathbb{Z})$. If $c = \sum a_\sigma \sigma$, let σ^* be any cell maximizing $\{f(\sigma) \mid a_\sigma \neq 0\}$. Then σ^* is critical.

Proof: Note that $c = \Phi(c) = \sum a_\sigma \Phi(\sigma)$. It follows that $a_{\sigma^*} = \langle c, \sigma^*\rangle = \sum a_\sigma \langle \Phi(\sigma), \sigma^*\rangle$. If $\sigma \neq \sigma^*$ and $f(\sigma) \le f(\sigma^*)$, then $\langle \Phi(\sigma),\sigma^*\rangle = 0$ (above prop). Thus, $0 \neq a_{\sigma^*} = a_{\sigma^*}\langle \Phi(\sigma^*),\sigma^*\rangle$ and so $\langle \Phi(\sigma^*),\sigma^*\rangle \neq 0$. But then the proposition above $\Rightarrow$ σ^* is critical.//

Prop: For N sufficiently large, $\Phi^N = \Phi^{N+1} = \cdots$

Proof: Let σ be a p-cell in X. Proceed by induction on $r = \#\{\tilde\sigma : f(\tilde\sigma) < f(\sigma)\}$. If $r = 0$, then we have $\Phi(\sigma) = \sigma$ or $\Phi(\sigma) = 0$. In either case, $\Phi^N(\sigma) = \Phi^{N+1}(\sigma) = \cdots$ for $N \ge 1$. For the inductive step suppose first that σ is not critical. Then $\Phi(\sigma) = \sum_{f(\tilde\sigma)<f(\sigma)} a_{\tilde\sigma}\tilde\sigma$. By induction, there is a sufficiently large N such that $\Phi^N(\tilde\sigma)$ is Φ-invariant when $f(\tilde\sigma) < f(\sigma)$. Then $\Phi^{N+1}(\sigma)$ is Φ-invariant.

If σ is not critical, let $c = V(\partial\sigma)$. Then $\Phi^n(\sigma) = \sigma + c + \Phi(c) + \cdots + \Phi^{n-1}(c)$. Thus, $\Phi^N(\sigma)$ is Φ-invariant $\Leftrightarrow$ $\Phi^N(c) = 0$ for some N. We know c is a sum of p-cells $\tilde\sigma$ with $f(\tilde\sigma) < f(\sigma)$ and so by induction there is an $\tilde N$ such that $\Phi^{\tilde N}(c)$ is Φ-invariant. Now, $c \in \mathrm{im}(V)$ and $\mathrm{im}(V)$ is Φ-invariant:
$\Phi V = (1+\partial V + V\partial)V = V(1+\partial V)$. It follows that $\Phi^{\tilde N}(c) \in \mathrm{im}(V)$. The proposition above tells us that $\mathrm{im}(V)$ is orthogonal to the critical cells; thus, $\Phi^{\tilde N}(c)$ is a Φ-invariant chain orthogonal to the critical cells + hence $\Phi^{\tilde N}(c) = 0$ by the previous lemma. We have thus found a large N with $\Phi^N(c) = 0$ + so $\Phi^N(c)$ is Φ-invariant.//

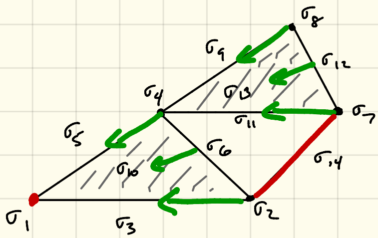
Since X Is A Finite Complex, There Is An N Such That For Every Chain c (Of Every Dimension) $\Phi^N(c) = \Phi^{N+1}(c) = \cdots$ Denote This Φ-Invariant Chain By $\Phi^\infty(c)$. Then For Every $p \geq 0$ We Have A Map $\Phi^\infty : C_p(X;\mathbb{Z}) \longrightarrow K_p(X;\mathbb{Z})$.

Thm: For Each $p \geq 0$, We Have An Isomorphism $H_p(K_\bullet(X;\mathbb{Z})) \cong H_p(X;\mathbb{Z})$.

Proof: Since $\Phi^\infty \circ i$ Is The Identity Map On $K_p(X;\mathbb{Z})$, We Have $id = \Phi^\infty_* \circ i_*$. We Must Show The Map $i_* \circ \Phi^\infty_*$ Is Also The Identity. For This We Need A Chain Homotopy $D : K_p(X;\mathbb{Z}) \to K_{p+1}(X;\mathbb{Z})$ Such That $id - i \circ \Phi^\infty = \partial D + D\partial$. Since $\Phi^\infty = \Phi^N$ For Some Large N,

$$\begin{aligned} id - i\circ\Phi^\infty &= id - \Phi^N = (id - \Phi)(id + \Phi + \cdots + \Phi^{N-1}) \\ &= (-\partial V - V\partial)(id + \Phi + \cdots + \Phi^{N-1}) \\ &= \partial\{-V(id + \Phi + \cdots + \Phi^{N-1})\} + \{-V(id + \Phi + \cdots + \Phi^{N-1})\}\partial \end{aligned}$$

We May Therefore Take $D = -V(id + \Phi + \cdots + \Phi^{N-1})$. //



Φ^∞:

$\sigma_1 \mapsto \sigma_1$, $\sigma_8 \mapsto \sigma_1$
$\sigma_2 \mapsto \sigma_1$, $\sigma_9 \mapsto 0$
$\sigma_3 \mapsto 0$, $\sigma_{10} \mapsto 0$
$\sigma_4 \mapsto \sigma_1$, $\sigma_{11} \mapsto 0$
$\sigma_5 \mapsto 0$, $\sigma_{12} \mapsto 0$
$\sigma_6 \mapsto 0$, $\sigma_{13} \mapsto 0$
$\sigma_7 \mapsto \sigma_1$, $\sigma_{14} \mapsto \sigma_3 + \sigma_5 + \sigma_{11} + \sigma_{14}$

Cycle In X

$$\begin{array}{lllllll} C_\bullet(X): & 0 \to \mathbb{Z}\{\sigma_{10},\sigma_{13}\} & \longrightarrow & \mathbb{Z}\left\{\begin{smallmatrix}\sigma_3,\sigma_5,\sigma_6\\ \sigma_9,\sigma_{11},\sigma_{12},\sigma_{14}\end{smallmatrix}\right\} & \to & \mathbb{Z}\left\{\begin{smallmatrix}\sigma_1,\sigma_2,\sigma_4\\ \sigma_7,\sigma_8\end{smallmatrix}\right\} & \to 0 \\ & & & \downarrow \Phi^\infty & & \downarrow \Phi^\infty & \\ K_\bullet(X): & 0 \to 0 & \longrightarrow & \mathbb{Z}\{\sigma_3+\sigma_5+\sigma_{11}+\sigma_{14}\} & \to & \mathbb{Z}\{\sigma_1\} & \to 0 \\ & & & \uparrow \Phi^\infty & & \uparrow \Phi^\infty & \\ M_\bullet(X): & 0 \to 0 & \longrightarrow & \mathbb{Z}\{\sigma_{14}\} & \longrightarrow & \mathbb{Z}\{\sigma_1\} & \to 0 \end{array}$$

For Each p, Set $M_p(X;\mathbb{Z})$ = Span Of Critical Cells In $C_p(X;\mathbb{Z})$. Then Φ^∞ Restricts To A Map $\Phi^\infty : M_p(X;\mathbb{Z}) \to K_p(X;\mathbb{Z})$. Note: If σ Is A Critical p-Cell, Then For Any Other Critical p-Cell $\tilde\sigma$ We Have $\langle \Phi^\infty(\sigma), \tilde\sigma \rangle = 0$. Indeed, $\Phi^\infty(\sigma) = \sigma + c$ For Some $c \in \operatorname{im}(V)$ And Since $\operatorname{im}(V)$ Is Orthogonal To $M_p(X;\mathbb{Z})$ The Result Follows.

Thm: The Map $\Phi^\infty : M_p(X;\mathbb{Z}) \to K_p(X;\mathbb{Z})$ Is An Isomorphism.

Proof. Let $c \in K_p(X;\mathbb{Z})$ And Consider The Chain $\tilde{c} = \sum_{\sigma \text{ critical}} \langle c,\sigma\rangle\sigma$. By The Remark Above, We Have $\langle \Phi^\infty(\tilde c), \sigma \rangle = \langle c, \sigma\rangle$. Thus, $\Phi^\infty(\tilde c) - c$ Is A Φ-Invariant Chain Such That For Any Critical Cell σ, $\langle \Phi^\infty(\tilde c) - c, \sigma\rangle = 0$. But Then $\Phi^\infty(\tilde c) - c = 0$ + So Φ^∞ Is Surjective. Now Suppose $\Phi^\infty(c) = 0$ For $c \in M_p(X;\mathbb{Z})$. Then For Any Critical Cell σ, $\langle \Phi^\infty(c), \sigma\rangle = 0$. The Remark Above Then Implies That For Any Critical Cell σ We Have $\langle c, \sigma \rangle = 0$ So That $c = 0$. Thus Φ^∞ Is Injective. //