Gradient-Like Vector Fields

Recall The Directional Derivative Of A Function $f: M \to \mathbb{R}$. Suppose $p \in M$ And $(x_1, \dots, x_n)$ Are Coordinates At p. If $v = \sum v_i \frac{\partial}{\partial x_i}$ Is A Tangent Vector At p, Then

$$v \cdot f = \sum_{j=1}^{n} v_j \frac{\partial f}{\partial x_j}(p)$$

Is The Derivative Of f In The Direction v. This Is A Real Number And $v \cdot f > 0$ If And Only If v Points In A Direction Where f Is Increasing.

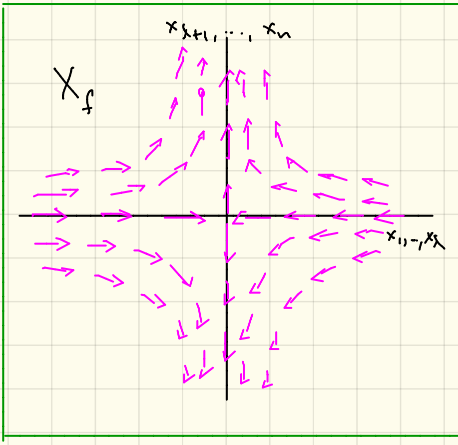
Now If U Is A Coordinate Nbhd Around p With Coordinates $(x_1, \dots, x_n)$, A Vector Field X on U Is Given By Choosing Smooth Functions $\xi_1, \dots, \xi_n$ And Setting

$$X = \sum_{j=1}^{n} \xi_j \frac{\partial}{\partial x_j}$$

For Example, If f Is A Smooth Function On U, Define A Vector Field X_f By

$$X_f = \frac{\partial f}{\partial x_1}\frac{\partial}{\partial x_1} + \cdots + \frac{\partial f}{\partial x_n}\frac{\partial}{\partial x_n}$$

X_f Is Called The Gradient Vector Field Of f.



Example: Consider $f = -x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2$

Then $X_f = -2x_1 \frac{\partial}{\partial x_1} - \cdots - 2x_\lambda \frac{\partial}{\partial x_\lambda} + 2x_{\lambda+1}\frac{\partial}{\partial x_{\lambda+1}} + \cdots + 2x_n \frac{\partial}{\partial x_n}$

A Vector Field Is Really A Differential Operator.

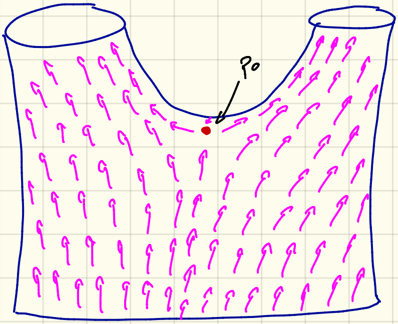
Note That
$$X_f \cdot f = \left(\sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\frac{\partial}{\partial x_j}\right) \cdot f$$
$$= \sum_{j=1}^{n} \left(\frac{\partial f}{\partial x_j}\right)^2$$
$$\geq 0$$

Note That $(X_f \cdot f)(p) > 0$ Unless p Is A Critical Point Of f.

Def: We Say That X Is A Gradient-Like Vector Field For The Morse Function $f: M \to \mathbb{R}$ If

1. $X \cdot f > 0$ Away From The Critical Points Of f
2. If p_0 Is A Critical Point Of Index λ, Then p_0 Has A Sufficiently Small Nbhd V with Coordinates $(x_1, \dots, x_n)$ Such That f Has Standard Form $f = -x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2$ And X Can Be Written As Its Gradient: $X = -2x_1\frac{\partial}{\partial x_1} - \cdots - 2x_\lambda \frac{\partial}{\partial x_\lambda} + 2x_{\lambda+1}\frac{\partial}{\partial x_{\lambda+1}} + \cdots + 2x_n \frac{\partial}{\partial x_n}$.

Note: If We Chose A Riemannian Metric On M, And Denote By $\langle X, Y \rangle$ The Inner Product Of Two Tangent Vectors Wrt This Metric, Then We Have $\langle X, X_f \rangle = X \cdot f$ For Any Vector Field X.



A Gradient-Like Vector Field

Thm: Let $f: M \to \mathbb{R}$ Be A Morse Function, M Compact. Then There Exists A Gradient-Like Vector Field For f.

Proof: (Sketch) Cover M By A Finite Number Of Coordinate Nbhds $U_1, \ldots, U_e$. We May Assume That Each Critical Point Has A Small Nbhd V Contained In A Compact Set $K_j \subset U_j$ For Exactly One j. Take X_j To Be The Standard Form Gradient Of f In U_j. Note $X_j \cdot f > 0$ Away From The Critical Points. Use Step Functions On The K_j To Glue The X_j Together Into A Globally Defined X. //

Integral Curves Suppose $c: \mathbb{R} \to M$ Is A Smooth Curve In M. Note That

$$\left\langle \frac{dc}{dt}, X_f \right\rangle = \frac{d(f \circ c)}{dt}$$

Given Any Vector Field X, We Say That c Is An Integral Curve For X If $\frac{dc}{dt}(t) = X(c(t))$; That Is, The Velocity Vector Of $c(t)$ At Time t Is Precisely The Vector $X(c(t))$. In Other Words, Integral Curves Are Flow Lines For Particles Moving In M With Velocity X. These Integral Curves Exist For All t (Lemma From p. 11).

In Particular, Consider X_f. If p Is Not Critical For f, Then Integral Curve $c_p(t)$ $(c_p(0) = p)$ Approaches Critical Points As $t \to \infty$ And $t \to -\infty$. The Velocity Vectors Approach 0, So We Never Actually Reach The Critical Points.

Homotopy Type And Critical Values

Recall That If $f: M \to \mathbb{R}$ Is Smooth, The Sublevel Set $M^a = \{x \in M \mid f(x) \leq a\}$ Is A Submanifold With Boundary $f^{-1}(a)$.

Thm: Let $f: M \to \mathbb{R}$ Be A Smooth Function. Let $a < b$ Be Such That $f^{-1}([a,b])$ Is Compact And Contains No Critical Points Of f. Then M^a Is Diffeomorphic To M^b. In Fact, M^a Is A Deformation Retract Of M^b So That The Inclusion $M^a \to M^b$ Is A Homotopy Equivalence.

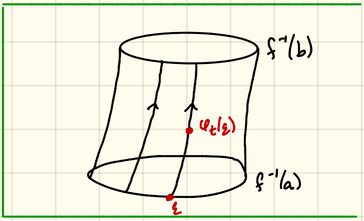
Proof: The One Line Proof Is "Flow Along The Integral Curves Of The Gradient Of f". More Precisely, Define $\rho: M \to \mathbb{R}$ To Be A Smooth Function Equal To $1/\langle X_f, X_f \rangle$ In The Compact Set $f^{-1}([a,b])$ And Which Vanishes Outside A Compact Nbhd Of This Set.

The Vector Field X Defined By $X_q = \rho(q)(X_f)_q$ Generates A 1-Parameter Family Of Diffeomorphisms $\varphi_t : M \to M$. For Fixed $q \in M$ Consider The Function $t \mapsto f(\varphi_t(q))$. If $\varphi_t(q)$ Lies In $f^{-1}([a,b])$, Then

$$\frac{d f(\varphi_t(q))}{dt} = \left\langle \frac{d\varphi_t(q)}{dt}, X_f \right\rangle = \langle X, X_f \rangle = 1.$$

Thus, The Map $t \mapsto f(\varphi_t(q))$ Is Linear With Derivative $+1$ As Long As $f(\varphi_t(q))$ Lies Btwn a And b.

Now Consider The Diffeomorphism $\varphi_{b-a} : M \to M$. This Carries M^a Diffeomorphically Onto M^b. This Proves The First Statement.



Define A 1-Parameter Family $r_t : M^b \to M^b$ By

$$r_t(q) = \begin{cases} q & \text{If } f(q) \le a \\ \varphi_{t(a-f(q))}(q) & \text{If } a \le f(q) \le b. \end{cases}$$

Then r_0 Is The Identity And r_1 Is A Retraction From M^b To M^a. //

<u>Note</u>: We've Actually Proved More: $f^{-1}([a,b])$ Is Diffeomorphic To $f^{-1}(a) \times [0,1]$. The Integral Curves $\varphi_t(q)$ Start A Height a At Time 0 And Proceed Linearly To Height b At Time $b-a$. Since $[0, b-a] \approx [0,1]$ The Result Follows.

<u>Thm</u>: Let $f : M \to \mathbb{R}$ Be A Morse Function With Critical Points $p_1, \dots, p_r$. Then There Is A Morse Function $f' : M \to \mathbb{R}$ Whose Critical Points Are $p_1, \dots, p_r$ Such That $f'(p_i) \neq f'(p_j)$ If $p_i \neq p_j$. Moreover, We Can Take f' As (C^2, ε)-Close To f As We Wish.

<u>Proof</u>: Suppose $f(p_1) = f(p_2) = c$. We May Choose Coordinates $(x_1, \dots, x_n)$ At p_1 And Write $f = -x_1^2 - \dots - x_\lambda^2 + x_{\lambda+1}^2 + \dots + x_n^2 + c$. Let X_f Be A Gradient Vector Field For f In This Coordinate System. Then

$$X_f \cdot f = \left(\frac{\partial f}{\partial x_1}\right)^2 + \dots + \left(\frac{\partial f}{\partial x_n}\right)^2$$
$$= 4\left(x_1^2 + \dots + x_\lambda^2 + \dots + x_n^2\right).$$

For Small ε, Consider The Discs D_ε And $D_{2\varepsilon}$ Centered At p_1. In The Region $D_{2\varepsilon} - \mathrm{int}(D_\varepsilon)$ We Have $4\varepsilon^2 \le X_f \cdot f \le 4(2\varepsilon)^2$. Denote By K The Compact Set D_ε And By U The Open Set $\mathrm{int}(D_{2\varepsilon})$; Let $h : U \to \mathbb{R}$ Be A Step Function For This Pair. Extend h To M By Setting $h = 0$ Outside U. Define $\tilde{f} = f + ah$ Where a Is Small. Since $f = \tilde{f}$ Outside U, f & $\tilde{f}$ Have The Same Critical Points There. Since $h = 1$ In $\mathrm{int}(D_\varepsilon)$ We See That p_1 Is The Only Critical Point Of Both f And $\tilde{f}$ There. So The Only Place Where They Might Have Different Critical Points Is The Region Btwn D_ε And $D_{2\varepsilon}$.

In This Region $\left|\frac{\partial f}{\partial x_i} - \frac{\partial \tilde{f}}{\partial x_i}\right| = \left|a \frac{\partial h}{\partial x_i}\right|$ $i = 1, \dots, n$. It Follows That If $|a|$ Is Small Enough We Can Make $\left|\sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\right)^2 - \sum_{j=1}^{n}\left(\frac{\partial \tilde{f}}{\partial x_j}\right)^2\right|$ Arbitrarily Small. The Function $\sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\right)^2$ Takes Minimum Value $4\varepsilon^2 > 0$ Btwn D_ε And $D_{2\varepsilon}$ And So If a Is Small Enough $\sum_{i=1}^{n}\left(\frac{\partial \tilde{f}}{\partial x_i}\right)^2$ Takes A Nonzero Minimum Value There. Thus $\tilde{f}$ Cannot Have A Critical Point In This Region + So f And $\tilde{f}$ Have The Same Sets Of Critical Points. Thus, $\tilde{f}$ Is A Morse Function On M. Also, $\tilde{f}(p_1) = f(p_1) + a$ And $\tilde{f}(p_2) = f(p_2)$.

Repeat This Argument For The Remaining Critical Points Of f To Yield The Required Function f'. It's Clear That At Each Stage We Can Choose The a And ε Small Enough To Ensure That f' Is (C^2, ε')-Close To f For Any Prescribed ε'. //

We Now Investigate What Happens When We Pass A Critical Point Of f. Suppose The Critical Points Of f Are $p_0, \dots, p_r$ With $f(p_0) < \dots < f(p_r)$. Set $c_i = f(p_i)$. Observe That $M^a = \phi$ If $a < c_0$. Also, If $c_n < a$, $M^a = M$.

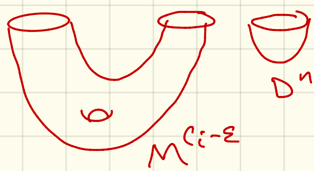
The Minimum c_0 Let $\varepsilon > 0$. Near p_0 We May Choose Coordinates $(x_1, \dots, x_n)$ So That $f = c_0 + x_1^2 + \dots + x_n^2$. The Index Of p_0 Must Be 0. Note That

$$M^{c_0+\varepsilon} = \{(x_1, \dots, x_n) \mid x_1^2 + \dots + x_n^2 \le \varepsilon\}$$

And This Is Diffeomorphic To The Disc D^n. We Visualize This As A "Bowl" Opening Upwards. If $n = 2$, It Looks Exactly Like That: $M^{c_0+\varepsilon} =$ 

If $m = 3$, We Have 3-Ball  Which Should Be Imagined As Curving "Up" In The 4th Dimension. In General, If c_i Is A Local Minimum (p_i Has Index 0) We Add An n-Disc D^n Curving Upward And $M^{c_i+\varepsilon} \simeq M^{c_i-\varepsilon} \amalg D^n$

An n-Disc Curving Upward Like This Is Called A **0-Handle** (More Precisely An n-Dim'l 0-Handle)

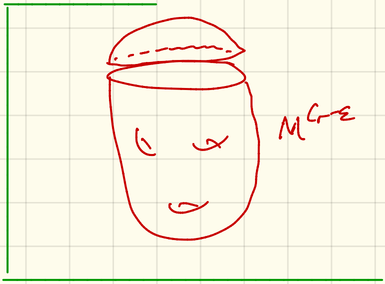


At The Opposite End, The Critical Value c_r Is A Maximum. In Coordinates Near p_r The Standard Form For f Is $f = c_r - x_1^2 - \dots - x_n^2$. The Index Of p_r Is n. If $c_r < a$, $M^a = M$. If $\varepsilon > 0$ Then We Can Express $M^{c_r-\varepsilon}$ As $x_1^2 + \dots + x_n^2 \ge \varepsilon$. This Corresponds To The Complement Of The n-Disc Of Radius $\sqrt{\varepsilon}$. As a Increases From $c_r - \varepsilon$ And Passes c_r We "Cap Off" The

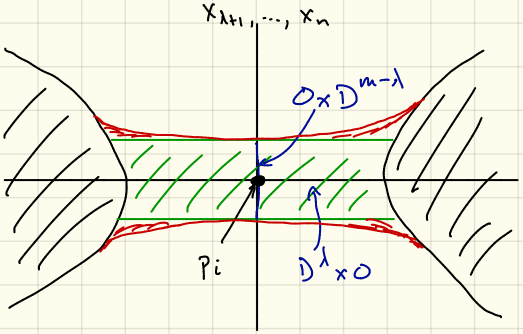
$M^{c_i-\varepsilon}$ With An n-Disc. That Is, We Attach An n-Disk Curving "Downward"

We Call This An n-Dimensional n-Handle.

In General, If c_i Is A Local Maximum Of f (p_i Has Index n) Then $M_{c_i-\varepsilon}$ Gets Capped Off By An n-Handle.



Now Suppose p_i Is A Critical Point Of Index λ, $0<\lambda<n$. In Local Coordinates Near p_i We Have $f = c_i - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2$. Let's Examine What The Manifold Looks Like Near p_i:



The Black Region Is $M^{c_i-\varepsilon}$ Which We Get By Setting $f \le c_i - \varepsilon$:

$$x_1^2 + \cdots + x_\lambda^2 - x_{\lambda+1}^2 - \cdots - x_n^2 \ge \varepsilon$$

The Green Area Corresponds To

$$x_1^2 + \cdots + x_\lambda^2 - x_{\lambda+1}^2 - \cdots - x_n^2 \le \varepsilon$$
$$x_{\lambda+1}^2 + \cdots + x_n^2 \le \delta$$

Where $\delta \ll \varepsilon$.

The Green Region Is An n-Dimensional Handle Of Index λ. It Is Diffeomorphic To $D^\lambda \times D^{n-\lambda}$. The λ-Disc $D^\lambda \times O = \{(x_1, \dots, x_\lambda, 0, \dots, 0) \mid x_1^2 + \cdots + x_\lambda^2 \le \varepsilon\}$ Is The Core Of The Handle And $O \times D^{n-\lambda} = \{(0, \dots, 0, x_{\lambda+1}, \dots, x_n) \mid x_{\lambda+1}^2 + \cdots + x_n^2 \le \delta\}$ Is The Co-Core.

Attach The λ-Handle $D^\lambda \times D^{n-\lambda}$ To $M^{c_i-\varepsilon}$ As Shown Above.

The Entire Shaded Region Is $M^{c_i+\varepsilon}$. Two Things Are True:

Thm: 1. $M^{c_i+\varepsilon}$ Is Diffeomorphic To $M^{c_i-\varepsilon} \cup D^\lambda \times D^{n-\lambda}$.

2. $M^{c_i+\varepsilon}$ Is Homotopy Equivalent To $M^{c_i-\varepsilon} \cup D^\lambda$.

Proof: Again, The One-Line Proof Is To Flow Along The Gradient Of f To Get A Deformation Retraction From $M^{c_i+\varepsilon}$ To $M^{c_i-\varepsilon} \cup D^\lambda$. The Details Are More Technical.

Note That $D^\lambda \cap M^{c_i-\varepsilon}$ Is Precisely The Boundary ∂D^λ So That D^λ Is Attached To $M^{c_i-\varepsilon}$ As Required. Construct A Smooth Function $F: M \to \mathbb{R}$ As Follows. Let $\mu: \mathbb{R} \to \mathbb{R}$ Satisfy (1) $\mu(0) > \varepsilon$; (2) $\mu(r) = 0$, $r \ge 2\varepsilon$; (3) $-1 < \mu'(r) \le 0$ For All r. Let F Coincide With f Outside The Coordinate Nbhd U And Let $F = f - \mu(x_1^2 + \cdots + x_\lambda^2 + 2x_{\lambda+1}^2 + \cdots + 2x_n^2)$ Inside U. For Convenience, Set $\xi, \eta: U \to [0, \infty)$ To Be $\xi = x_1^2 + \cdots + x_\lambda^2$; $\eta = x_{\lambda+1}^2 + \cdots + x_n^2$. Then $f = c_i - \xi + \eta$ And So $F(q) = c_i - \xi(q) + \eta(q) - \mu(\xi(q) + 2\eta(q))$ For $q \in U$.

Claim: $F^{-1}(-\infty, c_i+\varepsilon] = M^{c_i+\varepsilon}$.

Proof: Outside the ellipsoid $\xi + 2\eta \leq 2\varepsilon$ the functions f and F agree. Within this ellipsoid $F \leq f = c_i - \xi + \eta \leq c_i + \frac{1}{2}\xi + \eta \leq c_i + \varepsilon$. //

Claim: The critical points of F are the same as those of f.

Proof: Note that $\frac{\partial F}{\partial \xi} = -1 - \mu'(\xi + 2\eta) < 0$

$$\frac{\partial F}{\partial \eta} = 1 - 2\mu'(\xi + 2\eta) \geq 1$$

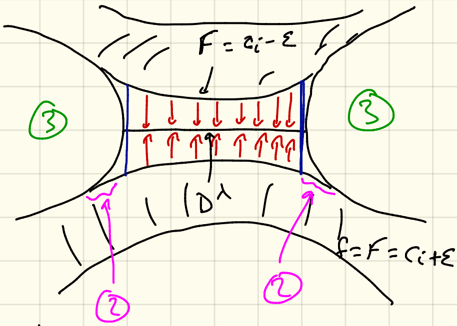
Since $dF = \frac{\partial F}{\partial \xi} d\xi + \frac{\partial F}{\partial \eta} d\eta$ and since $d\xi$ and $d\eta$ are simultaneously 0 at the origin, F has no critical points in U other than the origin. //

Consider the region $F^{-1}[c_i-\varepsilon, c_i+\varepsilon]$. The first claim (along with the fact that $F \leq f$) implies $F^{-1}[c_i-\varepsilon, c_i+\varepsilon] \subset f^{-1}[c_i-\varepsilon, c_i+\varepsilon]$ (and so is compact). The only possible critical point of F in this region is p_i. But $F(p_i) = c_i - \mu(0) < c_i - \varepsilon$. Thus $F^{-1}[c_i-\varepsilon, c_i+\varepsilon]$ contains no critical points! Thus: $F^{-1}(-\infty, c_i-\varepsilon]$ is a deformation retract of $M^{c_i+\varepsilon}$.

Denote this region by $M^{c_i-\varepsilon} \cup H$, where H is the closure of $F^{-1}(-\infty, c_i-\varepsilon] - M^{c_i-\varepsilon}$. (This is the green region - more or less). The cone D^λ consists of all points q with $\xi(q) \leq \varepsilon$, $\eta(q) = 0$; this is contained in H since $\frac{\partial F}{\partial \xi} < 0$ implies $F(q) \leq F(p_i) < c_i - \varepsilon$ but $f(q) \geq c_i - \varepsilon$ for $q \in D^\lambda$.

Claim: $M^{c_i-\varepsilon} \cup D^\lambda$ is a deformation retract of $M^{c_i-\varepsilon} \cup H$.

Proof: Here's a picture.



The arrows illustrate the retraction, but let's be careful. Define a family r_t: r_t = identity outside U. Inside U there are 3 cases.

(1) Here $\xi \leq \varepsilon$; $r_t(x_1, \dots, x_n) = (x_1, \dots, x_\lambda, t x_{\lambda+1}, \dots, t x_n)$.

Then $r_1 = \mathrm{id}$ and r_0 takes the whole region to D^λ. The fact that r_t takes $F^{-1}(-\infty, c_i - \varepsilon]$ to itself follows from $\frac{\partial F}{\partial \eta} > 0$.

(2) Here $\varepsilon \leq \xi \leq \eta + \varepsilon$; take $r_t(x_1, \dots, x_n) = (x_1, \dots, x_\lambda, s_t x_{\lambda+1}, \dots, s_t x_n)$ where $s_t \in [0,1]$ is defined by $s_t = t + (1-t)\left((\xi - \varepsilon)/\eta\right)^{1/2}$. $r_1 = \mathrm{id}$, r_0 maps the region into $f^{-1}(c_i - \varepsilon)$. Note that $s_t x_i$ are continuous as $\xi \to \varepsilon$, $\eta \to 0$.

(3) Here $\eta + \varepsilon \le \xi$ (i.e. We Are In $M^{c_i - \varepsilon}$). Let r_t Be The Identity. This Coincides With Previous Definition When $\xi = \eta + \varepsilon$.

This Completes The Proof. //

Thm: If $f: M \to \mathbb{R}$ Is A Morse Function And If Each M^a Is Compact, Then M Has The Homotopy Type Of A CW-Complex With One Cell Of Dimension λ For Each Crit. Pt. Of Index λ.

Proof: I Assume The Following Two Lemmas:

Lemma 1: Let $\varphi_0, \varphi_1: \partial e^\lambda \to X$ Be Homotopic Maps. Then $id: X \to X$ Extends To A Homotopy Equivalence $k: X \cup_{\varphi_0} e^\lambda \to X \cup_{\varphi_1} e^\lambda$.

Lemma 2: Let $\varphi: \partial e^\lambda \to X$ Be A Continuous Map. Then Any Homotopy Equivalence $f: X \to Y$ Extends To A Homotopy Equivalence $F: X \cup_\varphi e^\lambda \to Y \cup_{f\circ\varphi} e^\lambda$.

Now, Let $c_1 < c_2 < \dots$ Be The Critical Values Of f. Since Each M^a Is Compact This Sequence Has No Accumulation Points. If $a < c_1$ Then $M^a = \emptyset$. Suppose $a \ne c_1, c_2, \dots$ And That M^a Has The Homotopy Type Of A CW-Complex. Let $c = \min_i \{c_i \mid c_i > a\}$. For Sufficiently Small ε We Know (a) There Is A Homotopy Equivalence $h: M^{c-\varepsilon} \to M^a$ And (b) $M^{c+\varepsilon}$ Has The Homotopy Type Of $M^{c-\varepsilon} \cup_\varphi e^\lambda$ Where λ Is The Index Of The Critical Point p ($f(p) = c$). We've Assumed A Homotopy Equivalence $h': M^a \to K$, Where K Is A CW-Complex.

Now, $h' \circ h \circ \varphi$ Is Homotopic To A Cellular Map $\psi: \partial e^\lambda \to K^{(\lambda-1)}$ (The $(\lambda-1)$-Skeleton Of K). Then $K \cup_\psi e^\lambda$ Is A CW-Complex And Has The Same Homotopy Type As $M^{c+\varepsilon}$ (Use The Lemmas).

By Induction, It Follows That Each M^a Has The Homotopy Type Of A CW-Complex. If M Is Compact, This Completes The Proof. If All The Critical Points Lie In A Compact M^a, Then It's Easy To See That M^a Is A Deformation Retract Of M, So Again The Proof Is Complete.

If There Are Infinitely Many Critical Points, Then The Above Construction Gives A Sequence Of Homotopy Equivalences

$$\begin{array}{ccccc} M^{a_1} & \subset & M^{a_2} & \subset & \cdots \\ \downarrow & & \downarrow & & \\ K_1 & \subset & K_2 & \subset & \cdots \end{array}$$

Each Extending The Previous One. Let $K = \varinjlim K_i$ And Let $g: M \to K$ Be The Limit Map. Then g Induces Isomorphisms Of Homotopy Groups In All Dimensions. A Thm Of Whitehead Implies That g Is A Homotopy Equivalence. //

Examples

1. S^n $\quad f: S^n \to \mathbb{R}$, $f(x_1, \dots, x_{n+1}) = x_{n+1}$ Has Two Critical Points: $(0, \dots, 0, -1)$ Of Index 0 And $(0, \dots, 0, 1)$ Of Index n. If $-1 < a < 1$, Then $M^a \simeq D^n$ (The 0-Handle) And Then We Cap Off With An n-Handle.

Thm: Suppose M Is Compact And $f: M \to \mathbb{R}$ Is A Morse Function With Exactly Two Critical Points, Then M Is Homeomorphic To A Sphere.

Proof: The Two Critical Points Must Be The Minimum And The Maximum. Say $f(p)=0$ And $f(q)=1$ Are The Minimum And Maximum. If $\varepsilon > 0$ Is Small, The Sets $M^{\varepsilon} = f^{-1}[0,\varepsilon]$ And $f^{-1}[1-\varepsilon, 1]$ Are Both Closed n-Cells By The Morse Lemma. But M^{ε} Is Homeomorphic To $M^{1-\varepsilon}$ And So M Is The Union Of Two Closed n-Cells $M^{1-\varepsilon}$ And $f^{-1}[1-\varepsilon, 1]$, Matched Along Their Common Boundary.

2. $\mathbb{R}P^n$ Has A Morse Function $f: \mathbb{R}P^n \to \mathbb{R}$ With $n+1$ Critical Points Of Indices $0, 1, \dots, n$. So $\mathbb{R}P^n = e^0 \cup e^1 \cup \cdots \cup e^n$. The Attaching Maps Are Described As Follows. Note That $* = \mathbb{R}P^0 \subset \mathbb{R}P^1 \subset \mathbb{R}P^2 \subset \cdots \subset \mathbb{R}P^n$ And This Is Compatible With The Inclusions $S^0 \subset S^1 \subset S^2 \subset \cdots \subset S^n$. That Is, Under The Quotient Map $\pi_k : S^k \to \mathbb{R}P^k$ The Equator $S^{k-1} \subset S^k$ Maps To $\mathbb{R}P^{k-1} \subset \mathbb{R}P^k$. So The Attaching Map $\varphi_k : \partial e^k \to \mathbb{R}P^{k-1}$ Is Precisely The Double Cover $\pi_k : S^{k-1} \to \mathbb{R}P^{k-1}$ And Then $\mathbb{R}P^k = \mathbb{R}P^{k-1} \cup_{\pi_k} e^k$.

We Therefore Have The Chain Complex $C_*(\mathbb{R}P^n)$: $0 \to \underset{n}{\mathbb{Z}} \xrightarrow{1+(-1)^n} \underset{n-1}{\mathbb{Z}} \to \cdots \xrightarrow{2} \underset{1}{\mathbb{Z}} \xrightarrow{0} \underset{0}{\mathbb{Z}} \to 0$.

The Boundary Maps Are Most Easily Computed By Noting That $\partial e^k \to \mathbb{R}P^{k-1}/\mathbb{R}P^{k-2}$ Collapses The Equator And Identifies Antipodal Points. This Has Degree $1+(-1)^k$. So We Get

$$H_i(\mathbb{R}P^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i=0,\ i=n \text{ odd} \\ \mathbb{Z}/2 & i \text{ even } i<n \\ 0 & \text{else} \end{cases}$$

Note: Attaching Maps Matter. Since $\pi_1(S^1) = \mathbb{Z}$, There Are Infinitely Many Homotopy Classes Of Attaching Maps $\partial e^2 \to \mathbb{R}P^1$. We Only Get $\mathbb{R}P^2$ When We Take The Antipodal Map.

3. $\mathbb{C}P^n$ Has A Morse Function $f: \mathbb{C}P^n \to \mathbb{R}$ With $n+1$ Critical Points Of Indices $0, 2, 4, \dots, 2n$. It Follows That $\mathbb{C}P^n = e^0 \cup e^2 \cup \cdots e^{2n}$. Homology Is Easy To Calculate: $H_i(\mathbb{C}P^n;\mathbb{Z}) = \mathbb{Z}$, i Even $\leq 2n$ And 0 Otherwise. The Attaching Maps $\pi_k : \partial e^{2k} \to \mathbb{C}P^{k-1}$ Are The Hopf Fibrations $S^1 \to S^{2k-1} \to \mathbb{C}P^{k-1}$.

4. $T^2 = S^1 \times S^1$ Has A Morse Function With 4 Critical Points Of Indices $0, 1, 1, 2$. It Follows That $T^2 = e^0 \cup e^1 \cup e^1 \cup e^2$. The Attaching Maps $\partial e^1 \to e^0$ Are The Only Things They Could Be So That $e^0 \cup e^1 \cup e^1$ Is Simply A Wedge Of Two Circles $S^1 \vee S^1$. The 2-Cell Is Attached Via The Map $aba^{-1}b^{-1}$