HOMEWORK 1

Solve at least three and at most four of the following exercises. Please explain your work in a careful and logical way. Exercises with a star are longer and in some cases more complicated. At least one of the exercises you choose must have a star. If you plan on solving Exercise 8*, I then suggest you solve Exercise 9 as well. Finally, I recommend Exercise 3 to all you.

Exercise 1
Show that the paraboloid \( z = x^2 + y^2 \) is diffeomorphic to \( \mathbb{R}^2 \).

Exercise 2*
Show that the cylinder \( \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\} \) is a regular surface, and find parametrizations whose coordinates cover it. (In class we saw a single parametrization covering the whole cylinder minus a line!). Show that the cylinder is orientable. Compute the 2\textsuperscript{nd} Fundamental Form, Gauss Curvature, and Mean Curvature of the standard cylinder (Note that for this part of the exercise, you can simply work with the parametrization I gave in class). Finally, show that the cylinder cannot be locally isometric to the standard sphere of radius one \( S^2 \subset \mathbb{R}^3 \).

Exercise 3
In class, we defined the orientability of a regular surface \( S \subset \mathbb{R}^3 \) in terms of a family of coordinate neighborhoods covering \( S \), such that on the intersections of such coordinate neighborhoods the change of coordinates has positive Jacobian. Show that the orientability of a surfaces in \( \mathbb{R}^3 \) can alternatively be given in terms of the existence of a differentiable field of unit normal vectors. More precisely, show that a regular surface \( S \subset \mathbb{R}^3 \) is orientable if and only if there exists a differentiable field of unit normal vectors \( N : S \to \mathbb{R}^3 \) on \( S \).
Exercise 4

Let
\[ S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\} \]
be the standard unit sphere in \( \mathbb{R}^3 \). Show that any rotation of \( S^2 \) along the \( z \)-axis is a diffeomorphism of \( S^2 \) onto itself.

Exercise 5*

Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be a smooth function. Consider the regular surface \( S \subset \mathbb{R}^3 \) obtained by considering its graph in \( \mathbb{R}^3 \). Write down the standard parametrization and compute the 1\textsuperscript{st} Fundamental Form. Compute the differentiable field of unit normal vectors, say \( N : S \to \mathbb{R}^3 \), associated to the parametrization and compute the 2\textsuperscript{nd} Fundamental Form. Finally, write down the Gauss and Mean curvatures for such a graph.

Exercise 6

Let \( S_1 \) and \( S_2 \) be two regular surfaces, and let \( \varphi : S_1 \to S_2 \) be a diffeomorphism. Show that \( S_1 \) is orientable if and only if \( S_2 \) is orientable.

Exercise 7

Show that if a regular surface \( S \) contains an open set diffeomorphic to a Möbius strip, then \( S \) is nonorientable.

Exercise 8*

Let \( S \subset \mathbb{R}^3 \) be a regular surface which we assume to be oriented by the unit normal vector field \( N : S \to S^2 \). Given \( p_0 \in \mathbb{R}^3 \setminus S \), consider the smooth function \( f : S \to \mathbb{R} \) defined by
\[ f(p) = |p - p_0|^2 = \langle p - p_0, p - p_0 \rangle. \]
Show that \( p \in S \) is a critical point for \( f \) if and only if \( p - p_0 \in \text{Span}(N(p)) \). Finally, show that if \( p \) is a local maximum for \( f \) then the point \( p \) has to be an elliptic point for the surface \( S \).
Exercise 9 (Hint: Solve Ex. 8 first!)

Show that there is no compact regular surface $S \subset \mathbb{R}^3$ with non-positive Gauss Curvature everywhere. (You can assume the fact that any compact regular surface in $\mathbb{R}^3$ is orientable!)

Exercise 10*

The Catenoid is a regular surface in $\mathbb{R}^3$ given by the parametrization

$$X(u, v) = (c \cosh(v/c) \cos(u), c \cosh(v/c) \sin(u), v)$$

where $c$ is say a positive real constant, and where $u \in (0, 2\pi)$, $v \in \mathbb{R}$. Please compute the 1st and 2nd Fundamental Forms, the Gauss Curvature, and the Mean Curvature of the Catenoid. (The orientation being given by the standard unit normal vector field associated to the single parametrization given above!)

Exercise 11*

The so-called Theorema Egregium (Gauss) asserts that the Gauss Curvature of a surface is invariant under local isometries. Show that the “converse” of this theorem is not true. More precisely, verify that the surfaces

$$X(u, v) = (u \cos v, u \sin v, \log u),$$

$$\tilde{X}(u, v) = (u \cos v, u \sin v, v),$$

have equal Gauss Curvature at the points $X(u, v)$ and $\tilde{X}(u, v)$ but that the map $\tilde{X} \circ X^{-1}$ is not an isometry.