Solve at least three and at most four of the following exercises. Please explain your work in a careful and logical way.

Exercise 1

Let $M^m$ and $N^n$ be differentiable manifolds and let $\{(U_\alpha, X_\alpha)\}, \{(V_\beta, Y_\beta)\}$ be differentiable structures on $M$ and $N$ respectively. Show that the product space $M \times N$ is naturally a differentiable manifold of dimension $m + n$. Show also that the projections

$$\pi_1 : M \times N \to M, \quad \pi_2 : M \times N \to N,$$

are smooth maps which are also submersions.

Exercise 2

Show that the real projective space $\mathbb{P}^2_\mathbb{R}$ is not orientable. Hint: regard $\mathbb{P}^2_\mathbb{R}$ as a quotient of $S^2$, then find an open subset in $\mathbb{P}^2_\mathbb{R}$ diffeomorphic to a Möbius strip.

Exercise 3

Prove that the tangent bundle of ANY differentiable manifold is orientable. In particular $T\mathbb{P}^2_\mathbb{R}$ is an orientable manifold, even if $\mathbb{P}^2_\mathbb{R}$ is not orientable (cf. Exercise 2).

Exercise 4

Let $F : \mathbb{R}^3 \to \mathbb{R}^4$ be given by

$$F(x, y, z) = (x^2 - y^2, xy, xz, yz), \quad (x, y, z) = p \in \mathbb{R}^3.$$ 

Let $S^2 \subset \mathbb{R}^3$ be the unit sphere centered at the origin $0 \in \mathbb{R}^3$. Observe that $F(p) = F(-p)$ for any $p \in S^2$. We can then define the map $\tilde{\varphi} : \mathbb{P}^2_\mathbb{R} \to \mathbb{R}^4$ by setting

$$\tilde{\varphi}([p]) = F(p),$$
where \([p]\) is the point in \(\mathbb{P}^2_{\mathbb{R}}\) corresponding to the points \(\{p, -p\} \in S^2\). Prove that \(\tilde{\varphi}\) is an embedding.

**Exercise 5**

Let \(M_n(\mathbb{R}) = \mathbb{R}^{n^2}\) be the vector space of \(n \times n\) real valued matrices. Show that

\[
SL_n(\mathbb{R}) := \{ A \in M_n(\mathbb{R}) \mid \det A = 1 \}
\]

is a codimension one submanifold in \(M_n(\mathbb{R})\). Use this fact to prove that \(SL_n(\mathbb{R})\) is a Lie group of dimension \(n^2 - 1\).

**Exercise 6**

Let \(M\) be a non-compact manifold, and let \(X\) be a differentiable vector field over \(M\). Assume the support of \(X\) is contained in a compact subset of \(M\). Show that \(X\) is complete. Recall that the support of \(X\) is defined as

\[
\text{Supp}(X) := \{ p \in M \mid X(p) \neq 0 \in T_p M \}.
\]