Solve at least three and at most four of the following exercises. Please explain your work in a careful and logical way. **Exercise 1 is mandatory.**

**Exercise 1**

Let $(M, \langle \cdot, \cdot \rangle)$ be a Riemannian manifold. In class, we saw there is a *unique* symmetric affine connection $\nabla$ compatible with the metric $\langle \cdot, \cdot \rangle$. For any $X, Y, Z$ vector fields on $M$, this is given explicitly by the formula:

\[
\langle \nabla_X Y, Z \rangle = \frac{1}{2} \{ X \langle Y, Z \rangle + Y \langle Z, X \rangle - Z \langle X, Y \rangle - \langle [Y, X], Z \rangle - \langle [X, Z], Y \rangle - \langle [Y, Z], X \rangle \}.
\]

Show that the mapping

\[
\nabla : \chi(M) \times \chi(M) \to \chi(M)
\]

defined by this formula does indeed satisfy all of the axioms of an *affine connection.*

**Exercise 2**

In class, we have shown that the Christoffel symbols of the Levi-Civita connection of a Riemannian manifold $(M, g)$ are given in a coordinate neighborhood $X : U \to M$ by the neat formula

\[
\Gamma^m_{ij} = \frac{1}{2} \left\{ \frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ki} - \frac{\partial}{\partial x_k} g_{ij} \right\} g^{km},
\]

where we tacitly used Einstein summation convention. Now for regular surfaces $S \subset \mathbb{R}^3$ we also defined a set of Christoffel symbols which were given as solutions of certain linear systems given in terms of the 1st Fundamental form and its first derivatives. Show that if you set

\[ u = x_1, v = x_2, g_{11} = E, g_{22} = G, g_{12} = g_{21} = F, X_u = \frac{\partial}{\partial x_1}, X_v = \frac{\partial}{\partial x_2}, \]

and solve the linear systems defining the Christoffel symbols on a surface $S \subset \mathbb{R}^3$, you then recover the formula above.
Exercise 3

Let $S \subset \mathbb{R}^3$ be a regular surface equipped with the induced Riemannian metric (what we used to call the 1st Fundamental Form). Let $c : I \to S$ be a smooth curve, and let $V$ be a vector field tangent to $S$ along $c$. In other words, $V : I \to \mathbb{R}^3$ with $V(t) \in T_{c(t)}S$ for any $t \in I$. Show that $V$ is parallel if and only if $\frac{dV}{dt}$ is perpendicular to $T_{c(t)}S$ for any $t \in I$ (here $\frac{dV}{dt}$ is the usual derivative in $\mathbb{R}^3$).

Exercise 4

In Euclidean space, the parallel transport of a vector between two points does not depend on on the curve joining the two points. Show, by example, that this fact may not be true on arbitrary Riemannian manifold.

Exercise 5

Consider the upper half-plane

$$\mathbb{R}^2_+ := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 > 0\},$$
equipped with the metric given by $g_{11} = g_{22} = \frac{1}{x_2^2}$, $g_{12} = g_{21} = 0$ (this is the metric of the celebrated Lobatchevski’s non-euclidean geometry). Compute the Christoffel symbols of the Levi-Civita connections and verify they are given by:

$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^1_{22} = 0, \quad \Gamma^2_{11} = \frac{1}{x_2}, \quad \Gamma^1_{12} = \Gamma^2_{22} = -\frac{1}{x_2}.$$ 

Write down the geodesic equations on Lobatchevski’s plane. Can you find some solutions for some initial conditions of your choice?