ENERGY ESTIMATES FOR A CLASS OF SEMILINEAR ELLIPTIC EQUATIONS ON HALF EUCLIDEAN BALLS

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ABSTRACT. For a class of semi-linear elliptic equations with critical Sobolev exponents and boundary conditions, we prove point-wise estimates for blowup solutions and energy estimates. A special case of this class of equations is a locally defined prescribing scalar curvature and mean curvature type equation.

1. INTRODUCTION

In this article we consider

\[ \begin{cases} -\Delta u = g(u), & \text{in } B_3^+, \\ \frac{\partial u}{\partial x_n} = h(u), & \text{on } \partial B_3^+ \cap \partial \mathbb{R}^n_+, \end{cases} \tag{1.1} \]

where \( u > 0 \) is a positive continuous solution, \( B_3^+ \) is the upper half ball centered at the origin with radius 3, \( g \) is a continuous function on \((0, \infty)\) and \( h \) is locally Hölder continuous on \((0, \infty)\).

If \( g(s) = s^{\frac{n+2}{n-2}} \) and \( h(s) = cs^{\frac{n}{n-2}} \), the equation (1.1) is a typical prescribing curvature equation. If we use \( \delta \) to represent the Euclidean metric, then \( u^{\frac{n-2}{n}} \delta \) is conformal to \( \delta \). Equation (1.1) in this special case means the scalar curvature under the new metric is \( 4(n-1)/(n-2) \) and the boundary mean curvature under the new metric is \( -\frac{2}{n-2}c \). Equation (1.1) is very closely related to the well known Yamabe problem and the boundary Yamabe problem. For \( g \) and \( h \) we assume

\[ \begin{align*} GH_0 : & \quad g \text{ is a continuous function on } (0, \infty), \quad h \text{ is Hölder continuous on } (0, \infty), \\
\end{align*} \]

and

\[ \begin{align*} GH_1 : & \quad \lim_{s \to \infty} g(s)s^{\frac{n+2}{n-2}} \text{ is non-increasing, } \lim_{s \to \infty} g(s)s^{\frac{n+2}{n-2}} \in (0, \infty), \\
& \quad \lim_{s \to \infty} s^{-\frac{n}{n-2}}h(s) \text{ is non-decreasing and } \lim_{s \to \infty} s^{-\frac{n}{n-2}}h(s) < \infty. \\
\end{align*} \]

Let

\[ c_h := \lim_{s \to \infty} s^{-\frac{n}{n-2}}h(s). \tag{1.2} \]

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Then if \( c_h > 0 \) we assume
\[
GH_2 : \quad \sup_{0<s \leq 1} g(s)s^{-1} < \infty, \quad \text{and} \quad \sup_{0<s \leq 1} s^{-1}|h(s)| < \infty.
\]
If \( c_h \leq 0 \) our assumption on \( g, h \) is
\[
GH_3 : \quad \sup_{0<s \leq 1} g(s) < \infty, \quad \text{and} \quad \sup_{0<s \leq 1} |h(s)| < \infty.
\]
The main result of this article is concerned with the case \( c_h > 0 \):

**Theorem 1.1.** Let \( u > 0 \) be a solution of (1.1) where \( g \) and \( h \) satisfy \( GH_0 \) and \( GH_1 \). Suppose \( c_h > 0 \) and \( GH_2 \) also holds, then
\[
\int_{B^+_1} |
abla u|^2 + u^{2n-\frac{n}{2}} \leq C,
\]
for some \( C > 0 \) that depends only on \( g, h \) and \( n \).

Obviously if
\[
g(s) = c_1 s^{\frac{n+2}{n-2}}, \quad c_1 > 0 \quad \text{and} \quad h(s) = c_h s^{\frac{n}{n-2}}, \quad c_h > 0,
\]
g and \( h \) satisfy the assumptions in Theorem 1.1. The energy estimate (1.3) for this special case has been proved by Li-Zhang [9]. It is easy to see that the assumptions of \( g \) and \( h \) in Theorem 1.1 include a much larger class of functions. For example, for any non-increasing function \( c_1(s) \) satisfying \( \lim_{s \to \infty} c_1(s) > 0 \) and \( \lim_{s \to 0^+} c_1(s)s^{\frac{4}{n-2}} < \infty, \) \( g(s) = c_1(s)s^{\frac{n+2}{n-2}} \) satisfies the assumptions of \( g \). Similarly \( h(s) = c_2(s)s^{\frac{n}{n-2}} \) for a nondecreasing function \( c_2(s) \) with \( \lim_{s \to \infty} c_2(s) = c_h \) and \( \lim_{s \to 0^+} c_2(s)s^{\frac{2}{n-2}} < \infty \), satisfies the requirement of \( h \) in Theorem 1.1.

For the case \( c_h \leq 0 \) we have

**Theorem 1.2.** Let \( u > 0 \) be a solution of (1.1) where \( g \) and \( h \) satisfy \( GH_0 \) and \( GH_1 \). Suppose \( c_h \leq 0 \) and \( g, h \) satisfy \( GH_3 \), then the energy estimate (1.3) holds for \( C \) depending only on \( g, h \) and \( n \).

If we allow \( \lim_{s \to \infty} s^{-\frac{n+2}{n-2}} g(s) = 0 \), then the energy estimate (1.3) may not hold. For example, let \( g(s) = \frac{1}{4}(s+1)^{-2} \), then \( g \) satisfies the assumption in Theorem 1.2 except that \( \lim_{s \to \infty} s^{-\frac{n+2}{n-2}} g(s) = 0 \). Let \( u_j(x) = \sqrt{x_1 + j^2} - 1 \), it is easy to verify that \( u_j \) satisfies
\[
\begin{cases}
-\Delta u_j = g(u_j) & \text{in } B^+_3, \\
\frac{\partial u_j}{\partial x_n} = 0 & \text{on } \partial B^+_3 \cap \partial \mathbb{R}^n.
\end{cases}
\]
Note that \( h = 0 \) in this case. Then clearly (1.3) does not hold for \( u_j \).

The energy estimate (1.3) is closely related to the following Harnack type inequality:
\[
(\min_{B^+_1} u)(\max_{B^+_2} u) \leq C,
\]
which was proved by Li-Zhang [9] for the special case (1.4). Li-Zhang [9] also proved the (1.3) for (1.4) using (1.5) in their argument in an nontrivial way.
In the past two decades Harnack type inequalities similar to (1.5) have played an important role in blowup analysis for semilinear elliptic equations with critical Sobolev exponents. Pioneer works in this respect can be found in Schoen [13], Schoen-Zhang [14], Chen-Lin [2], and further extensive results can be found in [3, 5, 7, 9, 10, 12, 14, 16] and the references therein. Usually for a semi-linear equation without boundary condition, for example the conformal scalar curvature equation
\[
\Delta u + K(x)u^{\frac{n+2}{n-2}} = 0, \quad B_3,
\]
a Harnack inequality of the type
\[
(\max_{B_1} u)(\min_{B_2} u) \leq C
\]
immediately leads to the energy estimate
\[
\int_{B_1} |\nabla u|^2 + u^{\frac{2n}{n-2}} \leq C
\]
by the Green’s representation theorem and integration by parts. However, when the boundary condition as in (1.1) appears, using the Harnack inequality (1.5) to derive (1.3) is much more involved. In order to derive energy estimate (1.3) and pointwise estimates for blow up solutions, Li and Zhang prove the following results in [9]:

**Theorem A (Li-Zhang).** Let \( u > 0 \) be a solution of (1.1) where \( g \) and \( h \) satisfy \( GH_0, GH_1 \) and \( GH_3 \). Suppose in addition \( \max_{\overline{B}_1} \geq 1 \), then
\[
(\max_{\overline{B}_1} u)(\min_{\overline{B}_2} u) \leq C.
\]

Here we note that in Theorem A no sign of \( c_h \) is specified. One would expect the energy estimate (1.3) to follow directly from Li-Zhang’s theorem. This is indeed the case if \( c_h \leq 0 \). However for \( c_h > 0 \) substantially more estimates are needed in order to establish a precise point-wise estimate for blowup solutions. As a matter of fact we need to assume \( (GH_2) \) instead of \( (GH_3) \) in order to obtain (1.3).

The organization of this article is as follows. In section two we prove Theorem 1.1. The idea of the proof is as follows. First we use a selection process to locate regions in which the bubbling solutions look like global solutions. Then we consider the interaction of the bubbling regions. Using delicate blowup analysis and Pohozaev identity we prove that bubbling regions must be a positive distance apart. Then the energy estimate (1.3) follows. Even thought the main idea we use to prove Theorem 1.1 is similar to Li-Zhang’s proof of the special case \( g(s) = As^{\frac{n+2}{n-2}}, h(s) = c_h s^{\frac{n+2}{n-2}}, \) there are a lot of technical difficulties for the more general case. For example Li-Zhang’s proof relies heavily on the fact that the equation is invariant under scaling, thus they don’t need any classification theorem in their moving sphere argument. However in the more general case the equation is not scaling invariant any more and we have to use the classification theorem of Caffarelli-Gidas-Spruck or Li-Zhu. In section three we prove Theorem 1.2 using Theorem A and integration by parts.
2. Proof of Theorem 1.1

The proof of Theorem 1.1 is by way of contradiction. Suppose there is no energy bound, then there exists a sequence \( u_k \) such that

\[
\int_{B_1} |\nabla u_k|^2 + u_k^{\frac{2n}{n-2}} \to \infty.
\]  

We claim that \( \max_{B_{3/2}^+} u_k \to \infty \). Indeed, if this is not the case, which means there is a uniform bound for \( u_k \) on \( B_{3/2}^+ \), we just take a cut-off function \( \eta \in C^\infty \) such that \( \eta \equiv 1 \) on \( B_1^+ \) and \( \eta \equiv 0 \) on \( B_3^+ \setminus B_1^+ \) and \( |\nabla \eta| \leq C \). Multiplying the equation (1.1) by \( u_k \eta^2 \), using integration by parts and simple Cauchy’s inequality we obtain a uniform bound of \( \int_{B_1} |\nabla u_k|^2 \), a contradiction to (2.1).

**Definition 2.1.** Let \( \{u_k\} \) be a sequence of solutions of (1.1). Assume that \( x_k \to \tilde{x} \in B_2^- \) and \( \lim_{k \to \infty} u_k(x_k) = \infty \). If there exist \( C > 0 \) (independent of \( k \)) and \( \tilde{r} > 0 \) (independent of \( k \)) such that \( u_k(x)|x-\tilde{x}|^{2\tilde{r}} \leq C \) for \( |x-\tilde{x}| \leq \tilde{r} \), then we say that \( \tilde{x} \) is an isolated blow-up point of \( \{u_k\} \).

**Proposition 2.1.** Let \( \{u_k\} \) be a sequence of solutions of (1.1) and

\[
\max_{\bar{B}_1^+} u_k(x)|x|^{-\frac{2n}{n-2}} \to \infty \text{ as } k \to \infty,
\]

then there exist a sequence of local maximum points \( \bar{x}_k \) such that along a subsequence (still denoted as \( \{u_k\} \))

\[
v_k(y) := u_k(\bar{x}_k)^{-1} u_k(\bar{x}_k)^{-\frac{2}{n-2}} y + \bar{x}_k
\]

either converges uniformly over all compact subsets of \( \mathbb{R}^n \) to \( V \) that satisfies

\[
\Delta V + AV^{\frac{n+2}{n-2}} = 0, \quad \mathbb{R}^n
\]

(where \( A = \lim_{s \to \infty} g(s)s^{-\frac{2n}{n-2}} \)), or converges to \( V_1 \) defined on \( \{y \in \mathbb{R}^n; \quad y_n > -T\} \)

\[
(T := \lim_{k \to \infty} u_k(x_k)|x_k|^{\frac{n+2}{n-2}}, \quad \bar{x}_k)
\]

that satisfies

\[
\Delta V_1 + AV_1^{\frac{n+2}{n-2}} = 0, \quad \mathbb{R}^n \cap \{y_n > -T\},
\]

\[
\partial V_1/\partial y_n = c_h V_1^{\frac{n-2}{n+2}}, \quad y_n = -T.
\]

where \( c_h = \lim_{s \to \infty} s^{-\frac{n-2}{n+2}} h(s) \).

**Remark 2.1.** All solutions of (2.2) are described by the classification theorem of Caffarelli-Gidas-Spruck [1]. All solutions of (2.3) are described by Li-Zhu [11].

**Proof of Proposition 2.1:**

Let \( x_k \in B_1^+ \) be a sequence such that

\[
u_k(x_k)|x_k|^{-\frac{2n}{n-2}} \to \infty \text{ as } k \to \infty.
\]

Let \( d_k = |x_k| \) and \( S_k(y) = u_k(y)(d_k - |y - x_k|)^{\frac{n+2}{n-2}}, \forall y \in B_1^+ \). Set

\[
S_k(\bar{x}_k) = \max_{B(\bar{x}_k, d_k) \cap \{y > x_k\}} S_k.
\]
Once (2.9) is established, we see clearly that
\( M_{\sigma_k} \) over all compact subsets of \( \mathbb{R}^n \), then clearly (2.4) can be written as
\[
(2.5) \quad u_k(\tilde{x}_k) 2^{\frac{n+2}{2}} \sigma_k^{\frac{n+2}{2}} \geq u_k(x_k) d_k 2^{\frac{n+2}{2}} \to \infty \text{ as } k \to \infty.
\]
For all \( x \in B_{\sigma_k}^+(\tilde{x}_k) \), since
\[
(2.6) \quad u_k(x) (d_k - |x - x_k|) 2^{\frac{n+2}{2}} \leq u_k(\tilde{x}_k) (d_k - |x_k - \tilde{x}_k|) 2^{\frac{n+2}{2}},
\]
we have
\[
u_k(y) = M_k^{-1} u_k(M_k^{-\frac{2}{n+2}} y + \tilde{x}_k), \quad M_k^{-\frac{2}{n+2}} y + \tilde{x}_k \in B_j^+.
\]
Direct computation shows
\[
(2.7) \quad \Delta v_k(y) + (M_k v_k(y))^{-\frac{n+2}{2}} g(M_k v_k(y)) \cdot v_k(y) \frac{n+2}{2} = 0
\]
By (2.6) we have
\[
(2.8) \quad 0 \leq v_k(y) \leq 2^{\frac{n+2}{2}} \quad \forall y \in B(0, M_k^{-\frac{2}{n+2}} \sigma_k) \cap \{ y : n \geq -M_k^{-\frac{2}{n+2}} \tilde{x}_{kn} \}
\]
We consider the following two cases.

Case one: \( \lim_{k \to \infty} M_k^{-\frac{2}{n+2}} \tilde{x}_{kn} = \infty \).
Since \( M_k^{-\frac{2}{n+2}} \sigma_k \) and \( M_k^{-\frac{2}{n+2}} \tilde{x}_{nk} \) both tend to infinity, (2.7) is defined on \( |y| \leq l_k \) for some \( l_k \to \infty \). By (2.8) we assume that \( v_k \) is bounded above in \( B_{l_k} \). We claim that \( v_k \to V \) uniformly over all compact subsets of \( \mathbb{R}^n \) and \( V \) satisfies (2.2) with \( A = \lim_{s \to \infty} s^{-\frac{2}{n+2}} g(s) \). Indeed, we claim that for any \( R > 1 \),
\[
(2.9) \quad v_k(y) \geq C(R) > 0, \quad |y| \leq R.
\]
Once (2.9) is established, we see clearly that \( M_k v_k \to \infty \) over all \( B_R \), thus
\[
M_k^{-\frac{2}{n+2}} g(M_k v_k) = (M_k v_k)^{-\frac{n+2}{2}} g(M_k v_k) v_k^{\frac{n+2}{2}} \to AV^{\frac{n+2}{2}}
\]
over all compact subsets of \( \mathbb{R}^n \). Then it is easy to see that \( V \) solves (2.2).
Therefore we only need to establish (2.9) for fixed \( R > 1 \). Let
\[
\Omega_{R,k} := \{ y \in B_R : v_k(y) \leq 3 M_k^{-1} \}
\]
and
\[
\Omega_k(y) = M_k^{-\frac{2}{n+2}} g(M_k v_k) / v_k.
\]
It follows from \((GH_1)\) that in \(B_R \setminus \Omega_{R,k}\)
\[
a_k(y) \leq g(3) v_k^\frac{4}{n-2} \leq 4g(3).
\]
For \(y \in \Omega_{R,k}\) we use \((GH_2)\) to obtain
\[
a_k(y) \leq CM_k^{-\frac{4}{n-2}}, \quad y \in \Omega_{R,k}.
\]
In either case \(a_k(y)\) is a bounded function. From
\[
\Delta v_k(y) + a_k(y)v_k(y) = 0 \quad \text{in } B_R
\]
and standard Harnack inequality we have
\[
1 = v_k(0) \leq \max_{B_{R/2}} v_k \leq C(R) \min_{B_{R/2}} v_k.
\]
Thus (2.9) is established. Consequently \(V\), as the limit of \(v_k\) indeed solves (2.2). By the classification theorem of Caffarelli-Gidas-Spruck [1]
\[
V(y) = \left( \frac{n(n-2)}{A} \right)^{\frac{n-2}{4}} \left( \frac{\mu}{1 + \mu^2 |y|^2} \right)^{\frac{n-2}{4}}
\]
Obviously \(V\) has a maximum point \(\bar{x} \in \mathbb{R}^n\). Correspondingly there exists a sequence of local maximum points of \(u_k\), denoted \(\hat{x}_k\), that tends to \(\bar{x}\) after scaling. Thus \(v_k\) can be defined as in the statement of Proposition 2.1.

**Case two:** \(\lim_{k \to \infty} M_k^{-\frac{1}{2}} \hat{x}_kn < \infty\).

In this case we let
\[
T = \lim_{k \to \infty} M_k^{-\frac{1}{2}} \hat{x}_kn.
\]
It is easy to verify that \(v_k\) satisfies
\[
\begin{cases}
\Delta v_k(y) + (M_k v_k(y))^{-\frac{n-2}{2}} g(M_k v_k(y)) v_k^{\frac{n-2}{2}}(y) = 0, & \text{in } M_k^{-\frac{1}{2}} x + \hat{x}_k \in B_3^+,
\frac{\partial v_k}{\partial y_n} = (M_k v_k(y))^{-\frac{n-2}{2}} h(M_k v_k(y)) v_k^{\frac{n-2}{2}} v_k(y), & \text{on } y_n = -M_k^{-\frac{1}{2}} \hat{x}_kn.
\end{cases}
\]
We claim that for any \(R > 1\), there exists \(C(R) > 0\) such that
\[
(2.10) \quad v_k(y) \geq C(R) \text{ in } B_R \cap \{y_n \geq -M_k^{-\frac{1}{2}} \hat{x}_kn\}.
\]

The proof of (2.10) is similar to the interior case. Let \(T_k = M_k^{-\frac{1}{2}} x + \hat{x}_kn\) and \(p_k = (0', -T_k')\). On \(B(p_k, R) \cap \{y_n \geq -T_k\}\) we write the equation for \(v_k\) as
\[
\begin{cases}
\Delta v_k + a_k v_k = 0, & \text{in } B(p_k, R) \cap \{y_n \geq -T_k\},
\partial_n v_k + b_k v_k = 0, & \text{on } B(p_k, R) \cap \{y_n = -T_k\}.
\end{cases}
\]
(2.11)
where it is easy to use \(GH_2\) to prove that \(|a_k| + |b_k| \leq C\) for some \(C\) independent of \(k\) and \(R\). By a classical Harnack inequality with boundary terms (for example, see Lemma 6.2 of [15] or Han-Li [6]), we have
\[
1 = v_k(0) \leq \max_{B(p_k, R/2) \cap \{y_n \geq -T_k\}} v_k \leq C(R) \min_{B(p_k, R/2) \cap \{y_n \geq -T_k\}} v_k.
\]
Therefore \( v_k \) is bounded below by positive constants over all compact subsets. Thus the limit function \( V_1 \) solves (2.3). By Li-Zhu’s classification theorem [11],

\[
V_1(y) = \left( \frac{n(n-2)}{A} \right)^{\frac{1}{4}} \left( \frac{\lambda}{1 + \lambda^2 (|y'|^2 + |y_n - \tilde{y}_n|^2)} \right)^{\frac{1}{4}},
\]

where \( \tilde{y}_n = c_k \sqrt{(n-2)n/A}/((n-2)\lambda) \), \( \tilde{y} \in \mathbb{R}^{n-1} \), \( \lambda \) is determined by \( V_1(0) = 1 \). Thus the local maximum of \( V_1 \) can be used to defined \( v_k \) as in the statement of the proposition. Proposition 2.1 is established. \( \square \)

Proposition 2.1 determines the first point in the blowup set \( \Sigma_k \). The other points in \( \Sigma_k \) can be determined as follows: Consider the maximum of \( S_k(x) = u_k(x)^{\frac{n-2}{2}} \text{dist}(x, \Sigma_k) \).

If \( S_k(x) \) is uniformly bounded we stop. Otherwise the same selection process we get another blowup profile by either the classification theorem of Caffarelli-Gidas-Spruck or Li-Zhu. Eventually we have \( \{ q_{\theta}^i_k \} \in \Sigma_k \) (i = 1, 2, ...) that satisfy

\[
\begin{cases}
B_{r_i^j}^+(q_{\theta}^i_k) \cap B_{r_j^k}^+(q_{\theta}^j_k) = \emptyset, & \text{for } i \neq j, \\
|q_{\theta}^i_k - q_{\theta}^j_k|^{\frac{n-2}{2}} u_k(q_{\theta}^j_k) \to \infty, & \text{for } j > i, \\
u_k(x)^{\frac{n-2}{2}} \text{dist}(x, \Sigma_k) \leq C.
\end{cases}
\]

Take any \( q_k \in \Sigma_k \), let \( \sigma_k = \text{dist}(q_k, \Sigma_k \setminus \{ q_k \}) \) and we let

\[
\tilde{u}_k(y) = \sigma_k^{\frac{n-2}{2}} u_k(q_k + \sigma_k y), \quad \text{in } \Omega_k
\]

where \( \Omega_k := \{ y; \ q_k + \sigma_k y \in B_3^+ \} \). By the selection process we have

(2.12) \( \tilde{u}_k(y) \leq C|y|^{-\frac{n-2}{2}}, \quad |y| \leq 3/4, y \in \Omega_k \)

and

(2.13) \( \tilde{u}_k(0) \to \infty \).

We further prove in the following proposition that \( \tilde{u}_k \) decays like a harmonic function:

**Proposition 2.2.**

(2.14) \( \tilde{u}_k(0)|\tilde{u}_k(y)|^{n-2} \leq C, \quad \text{for } y \in B_{2/3} \cap \Omega_k \).

**Remark 2.2.** The meaning of Proposition 2.2 is each isolated blowup point is also isolated simple.
Proof. Direct computation shows that $\tilde{u}_k$ satisfies
\begin{equation}
\begin{cases}
\Delta \tilde{u}_k(y) + \sigma_k^{-\frac{n-2}{2}} g(\sigma_k^{-\frac{n-2}{2}} \tilde{u}_k) = 0, & \text{in } \Omega_k, \\
\partial_n \tilde{u}_k(y) = \sigma_k^{-\frac{n}{2}} h(\sigma_k^{-\frac{n}{2}} \tilde{u}_k), & \text{on } \partial \Omega_k \cap \{v_n = -\sigma_k^{-\frac{n}{2}} q_{kn}\},
\end{cases}
\end{equation}

Let $\tilde{M}_k = \tilde{u}_k(0)$. By (2.13) $\tilde{M}_k \to \infty$. Set
\[ v_k(z) = M_k^{-1} \tilde{u}_k(M_k^{-\frac{n}{2}} z), \quad \text{for } z \in \tilde{\Omega}_k, \]
where
\[ \tilde{\Omega}_k := \{z; \ |z| \leq M_k^{-\frac{n}{2}}, \ \tilde{M}_k^{-\frac{n}{2}} z \in \Omega_k\} \]
Note that $v_k$ is defined on a bigger set, but for the proof of Proposition 2.2 we only need to consider the part in $\Omega_k$.

Direct computation gives
\begin{equation}
\begin{cases}
\Delta v_k(z) + l_k^{-\frac{n-2}{2}} g(l_k v_k) = 0, & z \in \tilde{\Omega}_k, \\
\partial_v v_k = l_k^{-\frac{n}{2}} h(l_k v_k), & \{z_n = -T_k\} \cap \partial \tilde{\Omega}_k,
\end{cases}
\end{equation}
where $l_k = \sigma_k^{-\frac{n-2}{2}} \tilde{M}_k$ and $T_k = l_k^{-\frac{n}{2}} q_{kn}$. We consider two cases.

Case one: $T_k \to \infty$.

By the same argument as in the proof of Proposition 2.1, we know $v_k \to V$ in $C^{1,\alpha}_{loc}(\mathbb{R}^n)$ where $V$ solves (2.2). Thus there exist $R_k \to \infty$ such that
\[ \|v_k - V\|_{C^{1,\alpha}(B_k)} \leq CR_k^{-1}. \]
Clearly (2.14) holds for $|z| \leq \tilde{M}_k^{-\frac{n}{2}} R_k$, we just need to prove (2.14) for $|z| > \tilde{M}_k^{-\frac{n}{2}} R_k$. Since $V(0) = 1$ is the maximum point of $V$,
\[ V(z) = (1 + \frac{A}{n(n-2)} |z|^2)^{-\frac{n-2}{2}}, \quad z \in \mathbb{R}^n. \]

Lemma 2.1. There exists $k_0 > 1$ such that for all $k \geq k_0$ and $r \in (R_k, \tilde{M}_k^{-\frac{n}{2}})$,
\begin{equation}
\min_{\partial B_r \cap \tilde{\Omega}_k} v_k \leq 2 \left(\frac{n(n-2)}{A}\right)^{-\frac{n-2}{2}} r^{2-n}.
\end{equation}

Proof of Lemma 2.1:

Suppose (2.17) does not hold, then there exist $r_k$ such that
\begin{equation}
v_k(z) \geq 2 \left(\frac{n(n-2)}{A}\right)^{-\frac{n-2}{2}} r_k^{2-n}, \quad |z| = r_k, z \in \tilde{\Omega}_k.
\end{equation}
Clearly $r_k \geq R_k$.

Let
\[ v_k^\lambda(z) = \left(\frac{\lambda}{|z|}\right)^{n-2} v_k(z^\lambda), \quad z^\lambda = \frac{\lambda z}{|z|^2}. \]
The equation of $v_k^\lambda$, by direct computation, is
\begin{equation}
\Delta v_k^\lambda(z) + \left(\frac{\lambda}{|z|}\right)^{n+2} \frac{z \cdot z}{|z|^2} g\left(\frac{|z|}{\lambda}\right) v_k^\lambda(z) = 0, \quad \text{in } \Sigma_k
\end{equation}
where
\[ \Sigma_k := \{ z \in \Omega_k : |\lambda| < |z| < r_k \}. \]
Clearly $v_k^\lambda \to V^\lambda$ in $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^n)$ for fixed $\lambda > 0$. By direct computation
\begin{align*}
V(z) > V^\lambda(z), & \quad \text{for } \lambda \in (0,\left(\frac{n(n-2)}{A}\right)^{1/2}), \quad |z| > \lambda \\
V(z) < V^\lambda(z), & \quad \text{for } \lambda > \left(\frac{n(n-2)}{A}\right)^{1/2}, \quad |z| > \lambda.
\end{align*}
We shall apply the method of moving spheres for $\lambda \in \left(\frac{1}{2}\left(\frac{n(n-2)}{A}\right)^{1/2}, 2\left(\frac{n(n-2)}{A}\right)^{1/2}\right)$.
First we prove that for $\lambda_0 = \frac{1}{2}\left(\frac{n(n-2)}{A}\right)^{1/2}$,
\begin{equation}
\tag{2.20} v_k(z) > v_{k_0}^\lambda(z), \quad z \in \Sigma_k.
\end{equation}
To prove (2.20) we first observe that $v_k > v_{k_0}^\lambda$ in $B_R \setminus B_{\lambda_0}$ for any fixed $R$ large. Indeed, $v_k = v_k^{\lambda_0}$ on $\partial B_{\lambda_0}$. On $\partial B_{\lambda_0}$ we have $\partial_\nu V > \partial_\nu V^{\lambda_0}$. Thus the $C^{1,\alpha}$ convergence of $v_k$ to $V$ gives that $v_k > v_{k_0}^\lambda$ near $\partial B_{\lambda_0}$. Then by the uniform convergence we further know that (2.20) holds on $B_R \setminus B_{\lambda_0}$. On $\partial B_R$, we have
\begin{equation}
\tag{2.21} v_k(z) \geq \left(\frac{n(n-2)}{A}\right)^{\frac{n-2}{2}} - \varepsilon)|z|^{2-n}, \quad |z| = R
\end{equation}
and
\begin{equation}
\tag{2.22} v_{k_0}^\lambda(z) \leq \left(\frac{n(n-2)}{A}\right)^{\frac{n-2}{2}} - 2\varepsilon)|z|^{2-n}, \quad |z| \geq R
\end{equation}
for some $\varepsilon > 0$ independent of $k$. Next we shall use maximum principle to prove that
\begin{equation}
\tag{2.23} v_k(z) > \left(\frac{n(n-2)}{A}\right)^{\frac{n-2}{2}} - 2\varepsilon)|z|^{2-n} > v_{k_0}^\lambda(z), \quad z \in \Sigma_{\lambda_0} \setminus B_R.
\end{equation}
The proof of (2.23) is by contradiction. We shall compare $v_k$ and
\[ f_k := \left(\frac{n(n-2)}{A}\right)^{\frac{n-2}{2}} - 2\varepsilon)|z|^{2-n}. \]
Clearly $v_k - f_k$ is super harmonic in $\Sigma_{\lambda_0} - B_R$ and, by (2.21),(2.22) and (2.18), $v_k - f_k > 0$ on $\partial B_R$ and $\partial \Sigma_{\lambda_0} \cap (\mathbb{R}^n_+ \setminus B_R)$. If there exists $z_0 \in \partial \Sigma_{\lambda_0} \cap \{z_n = -T_k\}$ and
\begin{equation}
0 > v_k(z_0) - f_k(z_0) = \min_{\Sigma_{\lambda_0} \setminus B_R} v_k - f_k
\end{equation}
we would have
\begin{equation}
\tag{2.24} 0 < \partial_n(v_k - f_k)(z_0) = l_k^{n-2} h(l_k v_k(z_0)) - \partial_n f_k(z_0).
\end{equation}
Easy to verify $\bar{\partial} f_k (z_0) > N_k f_k (z_0)$ for some $N_k \to \infty$. However by $GH_1$

$$I_k \overset{\text{loc}}{=} h (l_k v_k (z_0)) \leq C v_k (z_0)$$

Easy to see it is impossible to have $v_k (z_0) < f_k (z_0)$ and (2.24). (2.23) is established. Before we employ the method of moving spheres we set

$$O_\lambda := \{ z \in \Sigma_\lambda : v_k (z) < \min \left( \left( \frac{|z|}{\lambda} \right)^{n-2}, 2 \right) v_k^2 (z) \}.$$

This is the region where maximum principle needs to be applied.

By $GH_1$ we have

$$\left( \frac{\lambda}{|z|} \right)^{n+2} I_k \overset{\text{loc}}{=} g (l_k (|z|/\lambda)^{n-2} v_k^2)$$

$$= \left( \left( \frac{|z|}{\lambda} \right)^{n-2} l_k v_k^2 \right)^{\frac{n-2}{n+2}} g \left( \left( \frac{|z|}{\lambda} \right)^{n-2} l_k v_k^2 (v_k^2)^{\frac{n-2}{n+2}} \right)$$

$$\leq (l_k v_k)^{\frac{n-4}{n+2}} g (l_k v_k) (v_k^2)^{\frac{n-2}{n+2}}, \text{ in } O_\lambda.$$

Therefore in $O_\lambda$ we have

$$\Delta v_k + (l_k v_k)^{\frac{n-4}{n+2}} g (l_k v_k) (v_k^2)^{\frac{n-2}{n+2}} \geq 0, \text{ in } O_\lambda.$$  

The equation for $v_k$ can certainly be written as

$$\Delta v_k + (l_k v_k)^{\frac{n-4}{n+2}} g (l_k v_k) (v_k^2)^{\frac{n-2}{n+2}} = 0.$$  

Let $w_{\lambda,k} = v_k - v_k^2$, we have, from (2.25) and (2.26)

$$\Delta w_{\lambda,k} + n (n-2) (l_k v_k)^{-\frac{2n-4}{n+2}} g (l_k v_k) \frac{1}{v_k^2} w_{\lambda,k} \leq 0, \text{ in } O_\lambda,$$

where $\xi_k$ is obtained from the mean value theorem.

Now we apply the method of moving spheres to $w_{\lambda,k}$. Let

$$\tilde{\lambda}_k = \inf \{ \lambda \in \left( \frac{n(n-2)}{A} \right)^{\frac{1}{2}} - \epsilon_0, \left( \frac{n(n-2)}{A} \right)^{\frac{1}{2}} + \epsilon_0 \} ; \text{ in } O_\lambda, \forall \mu > \lambda \}$$

where $\epsilon_0 > 0$ is chosen to be independent of $k$ and

$$v_k (z) > v^k (z), \quad |z| = r_k, \quad z \in \tilde{O}_k, \quad \forall \lambda \in [\lambda_0, \lambda_1].$$

From (2.18) we see that $\epsilon_0$ can be chosen easily. By (2.20), $\tilde{\lambda}_k > \left( \frac{n(n-2)}{A} \right)^{\frac{1}{2}} + \epsilon_0$.

We claim that $\tilde{\lambda}_k = \left( \frac{n(n-2)}{A} \right)^{\frac{1}{2}} + \epsilon_0$. Suppose this is not the case we have $\tilde{\lambda}_k < \left( \frac{n(n-2)}{A} \right)^{\frac{1}{2}} + \epsilon_0$. By continuity $w_{\lambda_k, k} \geq 0$ and by (2.18) $w_{\lambda_k, k} > 0$ on the outside boundary: $\partial \Sigma_{\lambda_k} \setminus (\partial B_{\lambda_k} \cup \{ z_n = -\bar{T}_{\lambda_k} \})$. By (2.27), if $\min_{x_k} w_{\lambda_k, k} = 0$, the minimum will have to appear on $\partial \Sigma_{\lambda_k} \setminus (\partial B_{\lambda_k} \cup \{ z_n = -\bar{T}_{\lambda_k} \})$. If there exists $x_0 \in \partial \Sigma_{\lambda_k} \cap \{ z_n = -\bar{T}_{\lambda_k} \}$, we have

$$0 < \partial w_{\lambda_k, k} (x_0)$$
Note that we have strict inequality because of Hopf Lemma. On the other hand, using $T_k \to \infty$ and elementary estimates we have
\[
\frac{\partial \bar{\lambda}_k}{\partial z_n} > N_k (\bar{\lambda}_k) \frac{\partial v_k}{\partial z_n}, \quad \text{in } O_{\bar{\lambda}_k} \cap \{z_n = -T_k\}, \quad \text{for some } N_k \to \infty.
\]
For $v_k$, $GH_1$ implies
\[
\frac{\partial v_k}{\partial z_n} \leq c_h v_k^{\frac{n}{n-2}}, \quad \text{in } O_{\bar{\lambda}_k} \cap \{z_n = -T_k\}
\]
where $c_h = \lim_{s \to \infty} s^{-\frac{2}{n-2}} h(s)$. Then it is easy to see that $w_{\bar{\lambda}_k} > 0$ on $\{z_n = -T_k\}$.

Then Hopf Lemma and the continuity lead to a contradiction of the definition of $\bar{\lambda}_k$. Thus we have proved $\bar{\lambda}_k = (\frac{m(n-2)}{A})^\frac{1}{2} + \epsilon_0$. However $V \in V^4$ for $|y| \lambda$ if $\lambda > (\frac{m(n-2)}{A})^\frac{1}{2}$. So it is impossible to have $\lim_{k \to \infty} \bar{\lambda}_k > (\frac{m(n-2)}{A})^\frac{1}{2}$. This contradiction proves (2.17) under Case one. Lemma 2.1 is established.

From Lemma 2.1 we further prove the spherical Harnack inequality for $v_k$. For fixed $k$, consider $2R_k \leq r \leq \frac{1}{2} \bar{\lambda}_k$ and let
\[
\bar{v}_k(z) = r^{\frac{n-2}{2}} v_k(rz).
\]
By (2.12), $\bar{v}_k(z) \leq C$. Direct computation yields
\[
\begin{cases}
\Delta \bar{v}_k(z) + r^{\frac{n-2}{2}} l_k^{-\frac{n-2}{2}} g(l_k r^{-\frac{n-2}{2}} \bar{v}_k) = 0, & \frac{1}{2} < |z| < 2, rz \in \bar{\Omega}_k, \\
\partial_z \bar{v}_k = r^{\frac{n-2}{2}} h(r^{-\frac{n-2}{2}} l_k \bar{v}_k), & \partial \bar{\Omega}_k.
\end{cases}
\]
Let
\[
a_k = r^{\frac{n-2}{2}} l_k^{-\frac{n-2}{2}} g(l_k r^{-\frac{n-2}{2}} \bar{v}_k) / \bar{v}_k \\
b_k = r^{\frac{n-2}{2}} h(r^{-\frac{n-2}{2}} l_k \bar{v}_k) / \bar{v}_k.
\]
By the definition of $l_k$ and $r$ we see that $r = o(1) l_k^{\frac{2}{n-2}}$. Using the assumptions of $g$ we have
\[
a_k(z) \leq \begin{cases}
g(1)^{\frac{2}{n-2}} \bar{v}_k \leq C, & \text{if } l_k r^{-\frac{n-2}{2}} \bar{v}_k(z) \geq 1, \\
Cr^2 l_k^{-\frac{n-2}{2}} = o(1), & \text{if } l_k r^{-\frac{n-2}{2}} \bar{v}_k(z) \leq 1,
\end{cases}
\]
and
\[
|b_k(z)| \leq \begin{cases}
c_h \bar{v}_k^\frac{2}{n-2} \leq C, & \text{if } l_k r^{-\frac{n-2}{2}} \bar{v}_k(z) \geq 1 \\
Cr l_k^\frac{2}{n-2} = o(1), & \text{if } l_k r^{-\frac{n-2}{2}} \bar{v}_k(z) \leq 1,
\end{cases}
\]
Hence $a_k$ and $b_k$ are both bounded functions.

Consequently the equation for $\bar{v}_k$ can be written as
\[
\begin{cases}
\Delta \bar{v}_k(z) + a_k \bar{v}_k = 0, & \frac{1}{2} < |z| < 2, rz \in \bar{\Omega}_k, \\
\partial \bar{v}_k = b_k \bar{v}_k, & \partial \bar{\Omega}_k \cap \{z_n = -T_k/r\}.
\end{cases}
\]
Clearly we apply classical Harnack inequality for two cases: either \( T_k/r > 1 \) or \( T_k/r \leq 1 \). In the first case we have
\[
\max_{|z|=3/4} \tilde{v}_k(z) \leq C \min_{|z|=3/4} \tilde{v}_k.
\]
In the second case we have
\[
\max_{|z|=1,\alpha \geq -T_k/r} \tilde{v}_k(z) \leq C \min_{|z|=1,\alpha \geq -T_k/r} \tilde{v}_k.
\]
Clearly (2.14) is implied. Proposition 2.2 is established for **Case one**.

**Case two:** \( \lim_{k \to \infty} T_k = T \)

Recall that \( v_k \) satisfies (2.16).

By the same argument as in the proof of Proposition 2.1, we know \( v_k \to V \) in \( C^{1,a}_{\text{loc}}(\mathbb{R}^n \cap \{ y_n \geq -T \}) \) where \( V \) solves (2.3). Thus, there exist \( R_k \to \infty \) such that
\[
\|v_k - V\|_{C^{1,a}(B_{R_k})} \leq CR_k^{-1}.
\]
Clearly (2.14) holds for \( |y| \leq \bar{M}_k^{2-n}R_k \cap \{ y_n \geq -T_k \} \), we just need to prove (2.14) for \( \{ |y| > \bar{M}_k^{2-n}R_k \} \cap \{ y_n \geq -T_k \} \). Since \( V(0) = 1 \) is the maximum point of \( V \),
\[
V(z) = (1 + \frac{A}{n(n-2)} |z|^2)^{-\frac{n-2}{2}}, \quad z \in \mathbb{R}^n.
\]

**Lemma 2.2.** There exists \( k_0 > 1 \) such that for all \( k \geq k_0 \) and \( r \in (R_k, \bar{M}_k^{2-n}) \),
\[
(2.29) \quad \min_{\partial B_r \cap \Omega_k} v_k \leq 2(\frac{n(n-2)}{A})^{-\frac{n-2}{2}} r^{2-n}.
\]

**Proof of Lemma 2.2:**

Just like the interior case suppose there exist \( r_k \geq R_k \) such that
\[
(2.30) \quad \min_{\partial B_{r_k} \cap \Omega_k} v_k > 2(\frac{n(n-2)}{A})^{-\frac{n-2}{2}} r_k^{2-n}.
\]

Let
\[
\tilde{v}_k(z) = v_k(z - T_k e_n), \quad \tilde{v}_k^2(z) = (\frac{\lambda}{|z|})^{n-2} \tilde{v}_k(z)
\]
and
\[
D_k := \{ z; \quad \bar{M}_k^{2-n}(z - T_k e_n) \in \Omega_k \cap B_{r_k} \}
\]
be the domain of \( \tilde{v}_k \). Then \( D_k \subset \mathbb{R}^n_+ \). Set
\[
\Sigma_k := \{ z \in D_k; \quad |z| > \lambda \}.
\]

Let \( \tilde{V} \) be the limit of \( \tilde{v}_k \) in \( C^{1,a}_{\text{loc}}(\mathbb{R}^n_+) \):
\[
\tilde{V}(z) = (1 + \frac{A}{n(n-2)} |z - T e_n|^2)^{-\frac{n-2}{2}},
\]
then there exists \( \lambda_0 < \lambda_1 \) depending only on \( n, A, T \) such that
\[
\tilde{V} > \tilde{V}^{\lambda_0} \quad \text{in} \quad \mathbb{R}^n_+ \setminus B_{\lambda_0}
\]
and

\[ \tilde{V} < V^{\lambda_1} \quad \text{in} \quad \mathbb{R}_+^n \setminus B_{\lambda_1}. \]

We shall employ the method of moving spheres to compare \( \tilde{v}_k \) and \( \tilde{v}_k^{\lambda} \) on \( \Sigma_\lambda \) for \( \lambda \in [\lambda_0, \lambda_1] \).

First we use the uniform convergence of \( \tilde{v}_k \) to \( \tilde{V} \) to assert that, for any fixed \( R > 1 \)

\[ (2.31) \quad \tilde{v}_k(y) > \tilde{v}_k^{\lambda_0}(y), \quad y \in \Sigma_{\lambda_0} \cap B_R. \]

For \( R \) large we have (let \( a_1 = \left( \frac{n(n-2)}{A} \right)^{\frac{2}{n-2}} \))

\[ \tilde{v}_k(y) \geq (a_1 - \varepsilon/5)|y|^{2-n} \quad \text{on} \quad \partial B_R \cap \mathbb{R}_+^n. \]

and

\[ \tilde{v}_k^{\lambda_0}(y) \leq (a_1 - 2\varepsilon/5)|y|^{2-n}, \quad |y| > \lambda_0. \]

To prove \( \tilde{v}_k > \tilde{v}_k^{\lambda_0} \) in \( \Sigma_{\lambda_0} \setminus B_R \) we compare \( \tilde{v}_k \) with \( w = (a_1 - 3\varepsilon/10)|y - A_1 e_n|^{2-n} \)

where \( A_1 = 1/(n-2)c_n a_1^{\frac{2}{n-2}} \). For \( R \) chosen sufficiently large we have

\[ w \geq \tilde{v}_k^{\lambda_0} \quad \text{in} \quad \Sigma_{\lambda_0} \setminus B_R \]

and

\[ \tilde{v}_k > w \quad \text{on} \quad \partial B_R \cap \Sigma_{\lambda_0}. \]

To compare \( \tilde{v}_k \) and \( w \) over \( \Sigma_{\lambda_0} \setminus B_R \), it is easy to see that \( \tilde{v}_k > w \) on \( \partial B_R \cap \Sigma_{\lambda_0} \) and \( \partial \Sigma_{\lambda_0} \setminus (B_R \cup \{z_n > 0\}) \). Since \( \tilde{v}_k - w \) is super-harmonic, the only thing we need to prove is on \( \partial \mathbb{R}_+^n \setminus B_{\lambda_0} \)

\[ (2.32) \quad \partial_n (\tilde{v}_k - w) < \xi_k (\tilde{v}_k - w), \quad z_n = 0. \]

for some positive function \( \xi_k \). Then standard maximum principle can be used to conclude that \( \tilde{v}_k > w_k \) on \( \Sigma_{\lambda_0} \setminus B_R \).

To obtain (2.32) first for \( \tilde{v}_k \) we use \( GH2 \) to have

\[ \partial_n \tilde{v}_k \leq c_h \tilde{v}_k^{\frac{n-2}{n}}, \quad z_n = 0. \]

On the other hand by the choice of \( A_1 \) we verify easily that

\[ \partial_n w > c_h w^{\frac{n-2}{n}}, \quad z_n = 0. \]

Thus (2.32) holds from mean value theorem. We have proved that the moving sphere process can start at \( \lambda = \lambda_0 \):

\[ \tilde{v}_k > \tilde{v}_k^{\lambda_0} \quad \text{in} \quad \Sigma_{\lambda_0}. \]

Let \( \bar{\lambda} \) be the critical moving sphere position:

\[ \bar{\lambda} := \min \{ \lambda \in [\lambda_0, \lambda_1]; \quad \tilde{v}_k^{\mu} > \tilde{v}_k^{\bar{\lambda}} \quad \text{in} \Sigma_{\mu}, \forall \mu > \lambda \}. \]

As in Case one we shall prove that \( \bar{\lambda} = \lambda_1 \), thus getting a contradiction from \( \tilde{V} < \tilde{V}^{\lambda_1} \) for \( |z| > \lambda_1 \). For this purpose we let

\[ w_{\lambda_1, k} = \tilde{v}_k - \tilde{v}_k^{\lambda_1}. \]
To derive the equation for \( w_{\lambda,k} \) we first recall from (2.16) and the definition of \( \tilde{v}_k \) that

\[
\begin{align*}
\Delta \tilde{v}_k(z) + I_k^{-\frac{n-2}{n}} g(l_k \tilde{v}_k) &= 0, \quad z \in \bar{\Omega}_k, \\
\frac{\partial \tilde{v}_k}{\partial n} &= I_k^{-\frac{n}{2}} h(l_k \tilde{v}_k), \quad \{z_n = 0\} \cap \partial \bar{\Omega}_k.
\end{align*}
\]

(2.33)

where \( I_k = \sigma_k^{-\frac{n+2}{n}} M_k \). Correspondingly \( \tilde{v}_k^\lambda \) satisfies

\[
\begin{align*}
\Delta \tilde{v}_k^\lambda + (\lambda \frac{|z|}{\lambda})^{n+2} I_k^{-\frac{n+2}{n}} g(l_k\frac{|z|}{\lambda})^{n-2} \tilde{v}_k^\lambda(z) &= 0, \quad \text{in } \bar{\Sigma}_\lambda, \\
\frac{\partial \tilde{v}_k^\lambda}{\partial n} &= (\lambda \frac{|z|}{\lambda})^n I_k^{-\frac{n}{2}} h(l_k\frac{|z|}{\lambda})^{n-2} \tilde{v}_k^\lambda(z), \quad \text{on } \partial \Sigma_\lambda \cap \{z_n = 0\}.
\end{align*}
\]

(2.34)

Let \( O_\lambda \) be defined as before. Then in \( O_\lambda \) we have, by \( GH_1 \),

\[
\begin{align*}
(\lambda \frac{|z|}{\lambda})^{n+2} I_k^{-\frac{n+2}{n}} g(l_k\frac{|z|}{\lambda})^{n-2} \tilde{v}_k^\lambda(z) &\leq (v_k l_k)^{-\frac{n-2}{n}} g(l_k v_k)(\tilde{v}_k^\lambda)^{\frac{n-2}{n}}, \quad \text{in } O_\lambda \\
(\lambda \frac{|z|}{\lambda})^n I_k^{-\frac{n}{2}} h(l_k\frac{|z|}{\lambda})^{n-2} \tilde{v}_k^\lambda(z) &\geq (l_k v_k)^{-\frac{n-2}{n}} h(l_k v_k)(\tilde{v}_k^\lambda)^{\frac{n-2}{n}}, \quad \text{on } \partial O_\lambda \cap \{z_n = 0\}.
\end{align*}
\]

The inequalities above yield

\[
\begin{align*}
\Delta w_{\lambda,k} + \xi_{1,k} w_{\lambda,k} &\leq 0, \quad \text{in } O_\lambda \\
\partial_n w_{\lambda,k} &\leq \xi_{2,k} w_{\lambda,k}, \quad \text{on } \partial O_\lambda \cap \{z_n = 0\}.
\end{align*}
\]

where \( \xi_{1,k} > 0 \) and \( \xi_{2,k} \) are continuous functions obtained from mean value theorem. It is easy to see that the moving sphere argument can be employed to prove that \( \tilde{\lambda} = \tilde{\lambda}_1 \), which leads to a contradiction from the limiting function \( \tilde{V} \). Thus Lemma 2.2 is established. □

Lemma 2.2 guarantees that on each radius \( R_k \leq r \leq \frac{1}{2} M_k \) the minimum of \( v_k \) is always comparable to \(|z|^{2-n}\). Re-scaling \( v_k \) as \( r^{\frac{n-2}{2}} v_k(rz) \) we see the spherical Harnack inequality holds by the \( GH_2 \) and \( GH_3 \). Thus Proposition 2.2 is established in Case Two as well.

□

Lemma 2.3. Let \( \{u_k\} \) be a sequence of solutions of (1.1) and \( q_k \to q \in B_1^+ \) be a sequence of points in \( \Sigma_k \). Then there exist \( C > 0, r_2 > 0 \) independent of \( k \) such that

\[
\begin{align*}
u_k(q_k)u_k(x) &\geq C |x - q_k|^{2-n} \text{ in } |x - q_k| \leq r_2, \quad x \in B_3^+.
\end{align*}
\]

Proof. Without loss of generality we assume that \( q_k \)s are local maximum points of \( u_k \). We consider two cases

Case one: \( u_k(q_k)^{\frac{1}{2}} q_kn \to \infty \).

Let \( M_k = u_k(q_k) \) and

\[
u_k(y) = M_k^{-\frac{n-2}{2}} \left( M_k^{-\frac{n-2}{2}} y + q_k \right), \quad y \in \Omega_k := \{y; M_k^{-\frac{n-2}{2}} y + q_k \in B_3^+\}.
\]
Then in this case \( v_k \) converges uniformly to
\[
V(y) = (1 + \frac{A}{n(n-2)} |y|)^{-\frac{n+2}{2}}
\]
over all compact subsets of \( \mathbb{R}^n \). For \( \varepsilon > 0 \) small we let
\[
\phi = (\frac{n(n-2)}{A} - \varepsilon) \frac{n+2}{2} |y|^{2-n} - M_k^2 |y| \leq M_k^2 \]
on \( y \geq R \) where \( R > 1 \) is chosen so that \( v_k > \phi \) on \( \partial B_R \). By direct computation we have
\[
\frac{\partial \phi}{\partial y_n} > N_k \phi \frac{n+2}{2}, \quad \text{on } y_n = -q_k n M_k
\]
for some \( N_k \to \infty \). It is easy to see that \( v_k \geq \phi \) on \( \partial \Omega_k \setminus \{ y_n = -M_k^2 q_k \} \). On \( \{ y_n = -M_k^2 q_k \} \) we have
\[
\partial_{y_n} (v_k - \phi) \leq c_h (v_k - \phi).
\]
Thus standard maximum principle implies \( v_k \geq \phi \) on \( \Omega_k \). Lemma 2.3 is established in this case.

Now we consider

**Case two:** \( M_k^{\frac{1}{2}} q_k \leq C \).

Let \( v_k \) be defined as in (2.35). In this case the boundary condition is written as
\[
\partial_{y_n} v_k = (M_k^{\frac{1}{2}} v_k) \frac{n+2}{2} h(M_k^{\frac{1}{2}} v_k) v_k^{\frac{n-2}{2}}, \quad y_n = -M_k \frac{1}{2} q_k.
\]
\( v_k \) converges to \( V_1 \) over all compact subsets of \( \mathbb{R}^n \{ y_n \geq -T \} \) where
\[
T = \lim_{k \to \infty} M_k^{\frac{1}{2}} q_k.
\]
\( V_1 \) satisfies (2.3).

For \( R \) large and \( \varepsilon > 0 \) small, both independent of \( k \), we have
\[
v_k(y) \geq (\frac{n(n-2)}{A} - \varepsilon) \frac{n+2}{2} |y|^{2-n}, \quad |y| = R.
\]
In \( B_R \cap \mathbb{R}^n_+ \) we have the uniform convergence of \( v_k \) to \( V_1 \). Our goal is to prove that \( v_k \) is bounded below by \( O(1) |y|^{2-n} \) outside \( B_R \). To this end let
\[
w(y) = (\frac{n(n-2)}{A} - 2 \varepsilon) \frac{n+2}{2} |y - A_1 e_n|^{2-n}
\]
where
\[
A_1 = c_h (\frac{n(n-2)}{A}) - T.
\]
Then it is easy to check that
\[
\frac{\partial w}{\partial y_n} > c_h w(y) \frac{n+2}{2}, \quad \text{on } y_n = -M_k^{\frac{1}{2}} q_k.
\]
By choosing \( R \) larger if needed we have
\[
v_k(y) > (\frac{n(n-2)}{A} - \varepsilon) \frac{n+2}{2} |y|^{2-n} > w(y), \quad |y| = R, \quad y \in \mathbb{R}^n_+.
\]
Then it is easy to apply maximum principle to prove \( v_k > w \) in \( \Omega_k \setminus B_R \). Lemma 2.3 is established.

Let \( q_k^1 \in \Sigma_k \) and \( q_k^2 \) be its nearest or almost nearest sequence in \( \Sigma_k \):
\[
|q_k^2 - q_k^1| = (1 + o(1))d(q_k^1, \Sigma_k \setminus \{q_k^1\}).
\]
We claim that

**Lemma 2.4.** There exists \( C > 0 \) independent of \( k \) such that
\[
\frac{1}{C} u_k(q_k^1) \leq u_k(q_k^2) \leq Cu_k(q_k^1).
\]

**Proof.** Let \( \sigma_k = d(q_k^1, \Sigma_k \setminus \{q_k^1\}) \) and
\[
\tilde{u}_k(y) = \sigma_k^{\frac{n+2}{2}} u_k(q_k^1 + \sigma_k y).
\]
We use \( e_k \) to denote the image of \( q_k^2 \) after scaling (so \( |e_k| \to 1 \)). Then in \( B_1, \tilde{u}_k(x) \sim \tilde{u}_k(0)^{-1} |x|^{2-n} \) for \( |x| \sim 1/2 \). On one hand, for \( |x| = \frac{1}{2} \) we have, by Lemma 2.3 applied to \( e_k \),
\[
\tilde{u}_k(0)^{-1} \left( \frac{1}{2} \right)^{2-n} \geq C \tilde{u}_k(e_k)^{-1}
\]
which is just \( u_k(q_k^2) \leq Cu_k(q_k^1) \). On the other hand, the same moving sphere argument can be applied to \( u_k \) near \( q_k^2 \) with no difference. The Harnack type inequality gives
\[
\max_{B(q_k^2, 1/4) \cap B_3^+} u_k \min_{B(q_k^2, 1/2) \cap B_3^+} u_k \leq C.
\]
Using
\[
\max_{B(q_k^2, 1/4) \cap B_3^+} u_k \geq u_k(q_k^2)
\]
and
\[
\min_{B(q_k^2, 1/2) \cap B_3^+} u_k \geq \min_{B(q_k^1, \sigma) \cap B_3^+} u_k
\]
we have
\[
(2.36) \quad \tilde{u}_k(e_k) \tilde{u}_k(0)^{-1} \leq C.
\]
Thus (2.36) gives \( u_k(q_k^2) \leq Cu_k(q_k^1) \). Lemma 2.4 is established.

**Remark 2.3.** Proposition 2.2 is not needed in the proof of Lemma 2.4.

The following lemma is concerned with Pohozaev identity that can be verified by direct computation.

**Lemma 2.5.** Let \( u \) solve
\[
\begin{cases}
\Delta u + g(u) = 0, & \text{in } B_\sigma^+,
\end{cases}
\]
\[
\partial_\nu u = h(u) \quad \text{on } \partial' B_\sigma^+.
\]
Then

\begin{equation}
(2.37) \quad \int_{\partial B_2^B} h(u)\left(\sum_{i=1}^{n-1} x_i \partial_i u + \frac{n-2}{2} u\right) + \int_{B_2^B} \left(nG(u) - \frac{n-2}{2} g(u)u\right) = \int_{\partial B_2^B} \left(\sigma(G) - \frac{1}{2} \nabla u^2 + (\partial_v u)^2\right) + \frac{n-2}{2} u \partial_v u
\end{equation}

where \( G(s) = \int_0^s g(t) \, dt \).

**Proposition 2.3.** There exists \( d > 0 \) independent of \( k \) such that

\[
\lim_{k \to \infty} |q_k^1 - q_k^2| \geq d.
\]

**Proof.** Recall that \( \sigma_k = (1 + o(1))|q_k^1 - q_k^2| \). We prove by way of contradiction. Suppose \( \sigma_k \to 0 \), et \( \tilde{M}_k = \tilde{u}_k(0) \). We claim that

\begin{equation}
(2.38) \quad \tilde{M}_k \tilde{u}_k(y) \to a|y|^{2-n} + b(y) \text{ in } C^2_{\text{loc}}(B_{3/4} \cap \tilde{\Omega}_k \setminus \{0\}), \quad a > 0, b(0) > 0
\end{equation}

where \( \tilde{\Omega}_k = \{y; \quad \sigma_k y + q_k^1 \in B_1^k\} \).

**Proof of (2.38):** As usual we consider the following two cases:

**Case one:** \( \lim_{k \to \infty} q_{1n}^k \tilde{M}_k^{\frac{2}{n-2}} \to \infty \), and **Case two:** \( \lim_{k \to \infty} q_{1n}^k \tilde{M}_k^{\frac{2}{n-2}} \to T < \infty \).

First we consider **Case one.** Let

\[
T_k = \tilde{M}_k^{\frac{2}{n-2}} q_{1n}^k.
\]

Recall the equation for \( \tilde{u}_k \) is (2.15). Multiplying \( \tilde{M}_k \) on both sides and letting \( k \to \infty \) it is easy to see from the assumptions of \( g \) and \( h \) that \( \tilde{M}_k \tilde{u}_k \to h \text{ in } C^2_{\text{loc}}(B_1 \setminus \{0\}) \) where \( h \) is a harmonic function defined in \( B_1 \setminus \{0\} \). Thus

\[
h(y) = a|y|^{2-n} + b(y)
\]

for some harmonic function \( b(y) \) in \( B_1 \). From the pointwise estimate in Lemma 2.3 we see that \( a > 0 \). Given any \( \epsilon > 0 \), we compare \( \tilde{u}_k \) and

\[
w_k := (a - \epsilon)(|y|^{2-n} - R^2_k)
\]

on \( |y| \leq R_k \). Here \( R_k \to \infty \) is less than \( T_k \). Observe that \( \tilde{u}_k > w_k \) on \( \partial B_{R_k} \) and \( |y| = \epsilon_1 \) for \( \epsilon_1 \) sufficiently small. Thus \( \tilde{u}_k > w_k \) by the maximum principle. Let \( k \to \infty \) we have, in \( B_1 \)

\[
a|y|^{2-n} + b(y) \geq (a - \epsilon)|y|^{2-n}, \quad B_1 \setminus B_{\epsilon_1}.
\]

Then let \( \epsilon \to 0 \), which implies \( \epsilon_1 \to 0 \) we have \( b(y) \geq 0 \) in \( B_1 \). Next we claim that \( b(0) > 0 \) because by Lemma 2.3 and Lemma 2.4 we have

\[
a|y|^{2-n} + b(y) \geq a_1|y - e|^{2-n} \quad \text{in } B_1
\]

for some \( a_1 > 0 \), where \( e = \lim_{k \to \infty} \epsilon_k \). Thus \( b(y) > 0 \) when \( y \) is close to \( e \), which leads to \( b(0) > 0 \). (2.38) is established in **Case one.**

**Case two.**

Again we first have \( \tilde{M}_k \tilde{u}_k \to h \text{ in } C^2_{\text{loc}}(B_1^kT \setminus \{0\}) \) and \( h \) is of the form

\[
h(y) = a|y|^{2-n} + b(y), \quad y_n \geq -T.
\]
To prove $b(y) \geq 0$ we compare, for fixed $\varepsilon > 0$, $\bar{M}_k \bar{u}_k$ with

$$w_k(y) = (a - \varepsilon)((|y - b_k e_n|^{2-n} - (R_k - 1)^{2-n})$$

where $b_k \to 0$ and $R_k \to \infty$ are chosen to satisfy

$$(n - 2)b_k R_k^{-2} > c_0 \sigma_k, \quad c_0 = \sup_{0 < t \leq 1} s|h(s)|$$

and

$$(n - 2)b_k > c_h \bar{M}_k^{-\frac{2}{n - 2}}. \varepsilon$$

It is easy to see that such $b_k$ and $R_k$ can be found easily. Let $h_k = \bar{M}_k \bar{u}_k$, and $\partial' \Omega_k = \partial \Omega_k \cap \{y_n = -T_k\}$. We divide $\partial' \Omega_k$ into two parts:

$$E_1 = \{z \in \partial' \Omega_k; \quad \bar{u}_k(z) \sigma_k^{-\frac{2}{n - 2}} \geq 1\}, \quad E_2 = \partial' \Omega_k \setminus E_1.$$ 

Then by the assumptions on $h$

$$\partial_n h_k \leq \begin{cases} c_0 \sigma_k h_k, & x \in E_2, \\ c_h \bar{M}_k^{-\frac{2}{n - 2}} h_k^\frac{2}{n - 2}, & x \in E_1, \end{cases}$$

With the choice of $b_k$ and $R_k$ it is easy to verify that

$$\partial_n w_k \geq \max\{c_0 \sigma_k w_k, c_h \bar{M}_k^{-\frac{2}{n - 2}} w_k^\frac{2}{n - 2}\} \text{ on } \partial' \Omega_k \cap B_{R_k}.$$

Thus standard maximum principle can be applied to prove $h_k \geq w_k$ on $\Omega_k \cap B_{R_k}$. Letting $k \to \infty$ first and $\varepsilon \to 0$ next we have $b(y) \geq 0$ in $B_1 \cap \{y_n \geq -T\}$. Then by Proposition 2.4 we see that $b(y) > 0$ when $y$ is close to $e$, the limit of $e_k$. The fact $\partial_n b = 0$ at 0 implies $b(0) > 0$. (2.38) is proved in both cases.

Finally to finish the proof of Proposition 2.3 we derive a contradiction from each of the following two cases:

**Case one:** $\lim_{k \to \infty} \bar{M}_k^{\frac{2}{n - 2}} q_{1n}^k > 0$.

In this case we use the following Pohozaev identity on $B_\sigma$ for $\sigma < \lim_{k \to \infty} \bar{M}_k^{\frac{2}{n - 2}} q_{1n}^k$:

$$\int_{B_\sigma} \left( nG_k(\bar{u}_k) - \frac{n - 2}{2} \bar{u}_k g_k(\bar{u}_k) \right)$$

$$= \int_{\partial B_\sigma} \left( \sigma (G_k(\bar{u}_k) - \frac{1}{2} |\nabla \bar{u}_k|^2 + |\partial_v \bar{u}_k|^2) + \frac{n - 2}{2} \bar{u}_k \partial_v \bar{u}_k \right)$$

where

$$g_k(s) = \sigma_k^{-\frac{2}{n - 2}} g(\sigma_k^{-\frac{2}{n - 2}} s), \quad G(t) = \int_0^t g(s) ds, \quad G_k(s) = \sigma_k^2 G(\sigma_k^{-\frac{2}{n - 2}} s).$$

First we claim that for $s > 0$,

$$G_k(s) \geq \frac{n - 2}{2n} sg_k(s).$$

(2.40)
Indeed, writing \( g(t) = c(t)t^\frac{n+2}{2} \), we see from \( GH_1 \) that \( c(t) \) is a non-increasing function, thus

\[
G_k(s) = \sigma_k^n G(\sigma_k^{-\frac{n+2}{2}} s) = \sigma_k^n \int_0^{\sigma_k^{-\frac{n+2}{2}} s} c(t)t^\frac{n+2}{2} dt
\]

\[
\geq \sigma_k^n c(\sigma_k^{-\frac{n+2}{2}} s) \int_0^{\sigma_k^{-\frac{n+2}{2}} s} t^\frac{n+2}{2} dt = \frac{n-2}{2n} c(\sigma_k^{-\frac{n+2}{2}} s)s^{\frac{2n}{n-2}} = \frac{n-2}{2n} s \bar{g}_k(s).
\]

Replacing \( s \) by \( \bar{u}_k \) we see that the left hand side of (2.39) is non-negative. Next we prove that

\[
\lim_{k \to \infty} \bar{M}_k^2 \int_{\partial B_\sigma} \left( \sigma(G_k(\bar{u}_k)) - \frac{1}{2} |\nabla \bar{u}_k|^2 + |\partial_{\nu} \bar{u}_k|^2 \right) < 0
\]

for \( \sigma > 0 \) small. Clearly after (2.41) is established we obtain a contradiction to (2.39). To this end first we prove that

\[
\bar{M}_k^2 G_k(\bar{u}_k) = o(1).
\]

Indeed, by \( GH_1 \) and \( GH_3 \)

\[
G_k(\bar{u}_k) = \sigma_k^n \int_0^{\sigma_k^{-\frac{n+2}{2}} \bar{u}_k} g(t) dt
\]

\[
\leq \begin{cases} 
\sigma_k^n \int_0^{\sigma_k^{-\frac{n+2}{2}} \bar{u}_k} c(t)dt, & \text{if } \sigma_k^{-\frac{n+2}{2}} \bar{M}_k^{-1} \leq 1, \\
\sigma_k^n \left( \int_0^1 c(t)dt + \int_1^{\sigma_k^{-\frac{n+2}{2}} \bar{u}_k} c(t) dt \right), & \text{if } \sigma_k^{-\frac{n+2}{2}} \bar{M}_k^{-1} > 1.
\end{cases}
\]

Therefore

\[
G_k(\bar{u}_k) \leq \begin{cases} 
C \sigma_k^2 \bar{M}_k^{-2}, & \text{if } \sigma_k^{-\frac{n+2}{2}} \bar{M}_k^{-1} \leq 1, \\
C \sigma_k^n + C \bar{M}_k^{\frac{2n}{n-2}}, & \text{if } \sigma_k^{-\frac{n+2}{2}} \bar{M}_k^{-1} > 1.
\end{cases}
\]

Clearly (2.42) holds in either case. Consequently we write the left hand side of (2.41) as

\[
\int_{\partial B_\sigma} \left( -\frac{1}{2} \sigma |\nabla h|^2 + \sigma |\partial_{\nu} h|^2 + \frac{n-2}{2} h \partial_{\nu} h \right) + o(1)
\]

where

\[
h(y) = a|y|^{2-n} + b(y), \quad b(0) > 0, a > 0.
\]

By direct computation we have

\[
\int_{\partial B_\sigma} \left( -\frac{1}{2} \sigma |\nabla h|^2 + \sigma |\partial_{\nu} h|^2 + \frac{n-2}{2} h \partial_{\nu} h \right)
\]

\[
= \int_{\partial B_\sigma} \left( -\frac{(n-2)^2}{a} b(0) \cdot \sigma^{1-n} + O(\sigma^{2-n}) \right) dS.
\]

Thus (2.41) is verified when \( \sigma > 0 \) is small.

The second case we consider is **Case two**: \( \lim_{k \to \infty} \bar{M}_k^\frac{2n}{n-2} q_{kn} = 0. \)
In this case we use the following Pohozaev identity on $B^+_\sigma$: Let

$$h_k(s) = \sigma_k^{\frac{n}{2}} h(\sigma_k^{-\frac{n-2}{2}} s),$$

then we have

$$(2.43) \quad \int_{\partial B^+_\sigma \cap \partial \mathbb{R}^n_{+}} h_k(\bar{u}_k)(\sum_{i=1}^{n-1} x_i \partial_i \bar{u}_k + \frac{n-2}{2} \bar{u}_k) + \int_{B^+_\sigma} (nG_k(\bar{u}_k) - \frac{n-2}{2} g_k(\bar{u}_k)\bar{u}_k)$$

$$= \int_{\partial B^+_\sigma \cap \partial \mathbb{R}^n_{+}} \left( \sigma(G_k(\bar{u}_k) - \frac{1}{2} |\nabla \bar{u}_k|^2 + (\partial_i \bar{u}_k)^2) + \frac{n-2}{2} \bar{u}_k \partial_i \nu \bar{u}_k \right)$$

Multiplying $\bar{M}_k^2$ on both sides and letting $k \to \infty$ we see by the same estimate as in Case one that the second term on the left hand side is non-negative, the right hand side is strictly negative. The only term we need to consider is

$$\lim_{k \to \infty} \frac{\bar{M}_k^2}{\bar{M}_k^2} \int_{\partial B^+_\sigma \cap \partial \mathbb{R}^n_{+}} h_k(\bar{u}_k)(\sum_{i=1}^{n-1} x_i \partial_i \bar{u}_k + \frac{n-2}{2} \bar{u}_k).$$

Let $H(s) = \int_{0}^{s} h(t)dt$, then from integration by parts we have

$$(2.44) \quad \int_{\partial B^+_\sigma \cap \partial \mathbb{R}^n_{+}} h_k(\bar{u}_k)(\sum_{i=1}^{n-1} x_i \partial_i \bar{u}_k + \frac{n-2}{2} \bar{u}_k)$$

$$= \int_{\partial B^+_\sigma \cap \partial \mathbb{R}^n_{+}} \sigma_k^{n-1} H(\sigma_k^{-\frac{n-2}{2}} \bar{u}_k) \sigma$$

$$+ \int_{\partial B^+_\sigma \cap \partial \mathbb{R}^n_{+}} (- (n-1) \sigma_k^{n-1} H(\sigma_k^{-\frac{n-2}{2}} \bar{u}_k) + \frac{n-2}{2} \bar{u}_k h_k(\bar{u}_k)) dx'.$$

For the first term on the right hand side of (2.44) we claim

$$(2.45) \quad \bar{M}_k^2 \sigma_k^{-1} H(\sigma_k^{-\frac{n-2}{2}} \bar{u}_k) = o(1) \quad \text{on } \partial B_\sigma.$$ 

Indeed, by $GH_1$ and $GH_2$

$$|H(\sigma_k^{-\frac{n-2}{2}} \bar{u}_k)| \leq \begin{cases} 
\int_{0}^{\sigma_k^{-\frac{n-2}{2}} \bar{u}_k} c dt, & \text{if } \sigma_k^{-\frac{n-2}{2}} \bar{u}_k \leq 1, \\
\int_{1}^{\sigma_k^{-\frac{n-2}{2}} \bar{u}_k} c dt + \int_{1}^{\sigma_k^{-\frac{n-2}{2}} \bar{u}_k} \sigma_k^{\frac{n-2}{2}} \bar{u}_k c \sigma^{\frac{n-2}{2}} dt, & \text{if } \sigma_k^{-\frac{n-2}{2}} \bar{u}_k > 1, 
\end{cases}$$

Using $\bar{u}_k = O(1/\bar{M}_k)$ on $\partial B_\sigma$ we then have

$$\bar{M}_k^2 \sigma_k^{-1} |H(\sigma_k^{-\frac{n-2}{2}} \bar{u}_k)| \leq \begin{cases} 
O(\sigma_k), & \text{if } \sigma_k^{-\frac{n-2}{2}} \bar{u}_k \leq 1, \\
O(\sigma_k) + O(\bar{M}_k^{\frac{n-2}{2}}), & \text{if } \sigma_k^{-\frac{n-2}{2}} \bar{u}_k > 1.
\end{cases}$$

Thus (2.45) is verified and the first term on the right hand side of (2.44) is $o(1)$.

Therefore we only need to estimate the last term of (2.44), which we claim is non-negative. Indeed, for $t > 0$, we write $h(t) = b(t)t^{\frac{n-2}{2}}$ for some non-decreasing
function \( b \). Then we have

\[
\sigma_k^{n-1} H(\sigma_k^{\frac{n-2}{2}} s) = \sigma_k^{n-1} \int_0^{\sigma_k^{\frac{n-2}{2}} s} h(t) dt
\]

\[
= \sigma_k^{n-1} \int_0^{\sigma_k^{\frac{n-2}{2}} s} b(t) t^{\frac{n-2}{2}} dt \leq \sigma_k^{n-1} b(\sigma_k^{\frac{n-2}{2}} s) \int_0^{\sigma_k^{\frac{n-2}{2}} s} t^{\frac{n-2}{2}} dt
\]

\[
= \frac{n-2}{2n-2} b(\sigma_k^{\frac{n-2}{2}} s) s^{\frac{n-2}{2}} = \frac{n-2}{2n-2} h_k(s) s.
\]

Replacing \( s \) by \( \tilde{u}_k \) in the above we see that the last term of (2.44) is non-negative. Thus there is a contradiction in (2.43) in Case two as well. Proposition 2.3 is established.

We are in the position to finish the proof of Theorem 1.1. By Proposition 2.3 there is a positive distance between any two members of \( \Sigma_k \). The uniform bound of \( \int_{B_3^+} u_k^{2n/n-2} \) follows readily from the pointwise estimates for blowup solutions near an isolated blowup point. The uniform bound for \( \int_{B_3^+} |\nabla u_k|^2 \) can be obtained by scaling and standard elliptic estimates for linear equations. Thus we have obtained a contradiction to (2.1). Theorem 1.1 is established.

3. PROOF OF THEOREM 1.2

If \( h \) is non-positive, the energy estimate follows from the Harnack inequality in a straight forward way. Indeed, let \( G(x, y) \) be a Green’s function on \( B_3^+ \) such that \( G(x, y) = 0 \) if \( x \in B_3^+, y \in \partial B_3^+ \cap \mathbb{R}^n_+ \) and \( \partial_n G(x, y) = 0 \) for \( x \in B_3^+, y \in \partial B_3^+ \cap \partial \mathbb{R}^n_+ \).

It is easy to see that \( G \) can be constructed by adding the standard Green’s function on \( B_3 \) its reflection over \( \partial \mathbb{R}^n_+ \). It is also immediate to observe that

\[
G(x, y) \geq C_n |x-y|^{2-n}, \quad x \in B_3^+, \quad y \in B_3^+.
\]

Multiplying \( G \) on both sides of (1.1) and integrating by parts, we have

\[
u(x) + \int_{\partial B_3^+ \cap \partial \mathbb{R}^n_+} h(u(y)) G(x, y) dS_y + \int_{\partial B_3^+ \cap \partial \mathbb{R}^n_+} u(y) \frac{\partial G(x, y)}{\partial \nu} dS_y
\]

\[
= \int_{B_3^+} g(u(y)) G(x, y) dy.
\]

Here \( \nu \) represents the outer normal vector of the domain. Using \( h \leq 0 \) and \( \partial_n G \leq 0 \), we have

\[
u(x) \geq \int_{B_3^+} g(u(y)) G(x, y) dy, \quad x \in B_3^+.
\]

In particular take let \( u(x_0) = \min_{\partial B_3^+} u \), then \( |x_0| = 2 \), thus

\[
C \geq \max_{B_3^+} u \cdot \min_{B_3^+} u \geq \int_{B_{3/2}^+} g(u(y)) u(y) G(x_0, y) dy \geq C \int_{B_{3/2}^+} g(u(y)) dy.
\]

Therefore we have obtained the bound on \( \int_{B_{3/2}^+} |\nabla u|^2 \). To obtain the bound on \( \int_{B_1^+} |\nabla u|^2 \), we use a cut-off function \( \eta \) which is 1 on \( B_1^+ \) and is 0 on \( B_2^+ \setminus B_{3/2}^+ \) and
\[ |\nabla \eta| \leq C. \] Multiplying \( u \eta^2 \) to both sides of (1.1) and using integration by parts and Cauchy inequality we obtain the desired bound on \( \int_{B^+_{\delta}} |\nabla u|^2 \). Theorem 1.2 is established. \( \Box \)

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