MGF1107 Notes

1 Logical fallacies

1. **Appeal to Popularity:** If P is a statement and many people believe it is true, then P is true.

   Example: "Ford makes the best pickup trucks in the world. More people drive Ford pickups than any other light truck."

2. **False Cause:** Statement P happens, then statement Q happens, so P must have caused Q.

   Example: "I put the quartz crystal on my forehead and in five minutes my headache was gone."

3. **Appeal to Ignorance:** There is no proof that P is true, therefore P is false.

   Example: "Scientists have not found any concrete evidence of aliens visiting Earth. Therefore, anyone who claims to have seen a UFO must be hallucinating."

4. **Hasty Generalization:** Statements P and Q are linked one time or a few times, therefore P causes Q or Q causes P.

   Example: "Two cases of childhood leukemia have occurred along the street where the high-voltage power lines run. The power lines must be the cause of the illnesses."

5. **Limited Choice:** Statement P is false, therefore statement Q must be true.

   Example: "You don't support the President, so you are not a patriotic American."

6. **Personal Attack:** I have a problem with the person or group claiming P is true, therefore P is not true.

   Example: Gwen - "You should stop drinking excessively because it's hurting your grades and it is unhealthy."

   Merle - "I've seen you drink a few too many on occasion yourself!"

7. **Circular Reasoning:** P is true because, P (possibly restated in different terms) is true.

   Example: "Society has an obligation to provide health insurance because health care is a right of citizenship."

8. **Diversion (Red Herring):** P is related to Q, and I have an argument concerning Q, therefore P is true.

   Example: "We should not continue to fund cloning research because there are so many ethical issues involved. Decisions are based on ethics, and we cannot afford to have too many ethical loose ends."

9. **Straw Man:** Using an argument for a distorted version of P to conclude that P is true.

   Example: Suppose the mayor of a large city proposes decriminalizing drug possession. His challenger in the upcoming election says, "The mayor doesn't think there's anything wrong with drug use, but I do."

2 Propositional Logic

Propositional logic is concerned with propositions involving statements and connectives. An example would be the statement "It is raining" or "If the sea level rises, then land in Florida will be under water."

Definition 1. of a propositional logic is a language, $\mathcal{L}$, that consists of:

(i) Letters $p_1, p_2, p_3, ...$

(ii) The connectives $\vee, \neg$

(iii) Grouping by parenthesis (and)

As an example, suppose the letters of our language, $\mathcal{L}$, are statements in English. For instance, p_1 could be the statement "It is raining."

Definition 2. The formulas of a propositional logic, $\mathcal{L}$, are as follows:

(i) Sentence letters are formulas

(ii) If A and B are formulas, then $(A \vee B)$ is a formula.

(iii) If A is a formula, then $\neg A$ is a formula.

Using the logic of English statements, let the statement "It is raining" be a letter, then $\neg$"It is raining" is the formula "It is not raining." If "It is sunny" and "It is cloudy" are letters, then ("It is sunny"$\vee$ "It is cloudy") are formulas.

The formulas in definition 2 are the only formulas of $\mathcal{L}$. If P is any formula of $\mathcal{L}$, then P is exactly one of (i),(ii), or (iii), and it is uniquely expressed as such. So if P is the formula $A \vee B$ then A and B are unique, and P is not a formula as in (i) or (iii) distinct from $A \vee B$.

Now that we have formed a grammar for a language, $\mathcal{L}$, we will give truth interpretations to formulas of $\mathcal{L}$ to give the language meaning.

Definition 3. A truth interpretation of a language, $\mathcal{L}$ is a function, I, such that for any formula of $\mathcal{L}$, I is defined as follows:

(i) If P is a letter, then
$$I(P) \in \{0, 1\}$$

(ii) If P is a formula, then
$$I(\neg P) = 1 - I(P)$$

(iii) If $A \vee B$ is a formula, then
$$I(A \vee B) = \max\{I(A), I(B)\}$$

Truth tables can be used to determine the value of I for a given formula. Examples are given below.

P	$\neg P$
1	0
0	1

P	Q	$(P \vee Q)$
1	1	1
1	0	1
0	1	1
0	0	0

Two formulas are logically equivalent if they have they both imply each other, which can be shown by constructing truth tables. If the truth values for two statements match, then they are logically equivalent. Using the two connectives, $\neg, \vee$, formulas can be given for other common connectives such as the conditional, $\Rightarrow$, and 'and', $\wedge$. An example that gives a formula for $\rightarrow$ is shown below.

P	Q	$\neg P$	$(\neg P \vee Q)$	$(P \Rightarrow Q)$
1	1	0	1	1
1	0	0	0	0
0	1	1	1	1
0	0	1	1	1

Using truth tables, we can show how the negation symbol, $\neg$, distributes over $\wedge$ and $\vee$

P	Q	$(P \vee Q)$	$\neg(P \vee Q)$	$\neg P$	$\neg Q$	$(\neg P \wedge \neg Q)$
1	1	1	0	0	0	0
1	0	1	0	0	1	0
0	1	1	0	1	0	0
0	0	0	1	1	1	1

P	Q	$(P \wedge Q)$	$\neg(P \wedge Q)$	$\neg P$	$\neg Q$	$(\neg P \vee \neg Q)$
1	1	1	0	0	0	0
1	0	0	1	0	1	1
0	1	0	1	1	0	1
0	0	0	1	1	1	1

Therefore, we have the following equivalences:

1. $\neg P \vee Q = P \Rightarrow Q$
2. $\neg(P \vee Q) = \neg P \wedge \neg Q$
3. $\neg(P \wedge Q) = \neg P \vee \neg Q$

Exercise 1. Find more simple formulas for the following:

1. $\neg(P \vee \neg Q)$
2. $\neg(Q \wedge \neg P) \vee P$
3. $\neg(\neg P \vee Q) \vee (P \wedge \neg R)$

3 Sets

Definition 4. A set is a collection of objects, called elements. The elements of a set are listed in curly brackets.

Example: $\mathbb{N} = \{0, 1, 2, ...\}$ is the set of natural numbers. $\mathbb{Z} = \{0, \pm 1, \pm 2, ...\}$ is the set of integers.

If S is a set, and x belongs to that set, we write $x \in S$ to say x is an element of S.

To avoid listing the elements of a set, we can define a set explicitly as follows:

$$S = \{x \mid x \text{ is a positive integer}\}$$

which reads S is the set of all x such that x is a positive integer.

Example: $S = \{x \in \mathbb{R} \mid x < 9\}$ is the set of all real numbers less than 9. S is the truth set of the statement $x < 9$, the set of numbers that x represents for which $x < 9$ is true.

Now we will define some operations and relations involving sets.

Definition 5. Let A and B be sets. The intersection of A and B is

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

The intersection of A and B is the set of elements that are in both A and B. Drawing a Venn diagram of A and B, $A \cap B$ would be the region in which A and B overlap.

Definition 6. Let A and B be sets. The union of A and B is

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

The union of A and B is the set of elements that are either in A or in B. So on a Venn diagram, $A \cup B$ is the region of A along with the region of B.

Definition 7. Let A and B be sets. A set minus B is the set

$$A \setminus B = \{x \mid x \in A \wedge x \notin B\}$$

Definition 8. Let A and B be sets. We say A is a subset of B, $A \subseteq B$ if for every $x \in A$, x is also in B. Logically,

$$x \in A \Rightarrow x \in B$$

$A \setminus B$ is the set of elements of A "minus" the elements of B.

Exercise 2. Let $A = \{x \mid x < 5\}$, $B = \{x \mid x > 9\}$, and $C = \{x \mid 1 < x < 12\}$. Use a Venn diagram to illustrate the following, and write them in set notation:

1. $A \cap B$
2. $A \cup B$
3. $C \cap B$
4. $(C \cup A) \setminus B$
5. $(C \cup B) \cap A$
6. $C \setminus (A \cup B)$
7. $(A \cup B) \cap C$

Exercise 3. Show the following relations using a Venn diagram:

1. $(A \cup B) \setminus C = (A \cup B) \setminus (C \setminus A)$
2. $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$

4 Quantifiers

To express truth of statements with variables, $P(x)$, we introduce quantifiers. The set of all possible values of x, U, is called the *universe of discourse*. If $P(x)$ is true for every element in the universe of discourse, then we write $\forall x \in U$, $P(x)$. The symbol $\forall$ reads "for all". $\forall$ is the universal quantifier. The statement $\forall x \in U$, $P(x)$ reads "For all x in the universe of discourse, U, $P(x)$." To say "there exists an x such that $P(x)$", we write $\exists x$, $P(x)$. The symbol, $\exists$, is the existential quantifier, which reads "there exists". This is saying that there is at least one value for which $P(x)$ is true.

Example: Let $P(x)$ be the statement $x^2 \geq 0$, and the universe of discourse is the set of real numbers, $\mathbb{R}$. Since the square of every real number is non-negative, we have $\forall x$, $x^2 \geq 0$.

Example: Let $P(x)$ be the statement $x < 9$, and the universe of discourse is $\mathbb{R}$. Since $8 \in \mathbb{R}$ and $8 < 9$, we know that $\exists x \in \mathbb{R}$ such that $x < 9$.

Example: Analyze the logical form of the statement "If anyone in the dorm has a friend who has the measles, then everyone in the dorm will have to be quarantined." In this example, "anyone" means we will use the existential quantifier. We could easily "If anyone..." with "If there is at least one person..." Now let

1. $D(x)$ be the statement "x is in the dorm"
2. $F(x, y)$ be "x is a friend of y"
3. $M(x)$ be x has the measles
4. $Q(x)$ be x will have to be quarantined.

We can use these to write the statement "x is in the dorm and x has a friend, y, with measles" as "$\exists x\Big(D(x) \wedge \exists y\big(F(x, y) \wedge M(y)\big)\Big)$". We can write "everyone in the dorm will have to be quarantined." as "$\forall z(D(z) \Rightarrow Q(z)$." Putting the two together as if-then, have

$$\exists x\Big(D(x) \wedge \exists y\big(F(x, y) \wedge M(y)\big)\Big) \Rightarrow \forall z\Big(D(z) \Rightarrow Q(z)\Big)$$

In this example, we could have defined the set D =the set of all people in the dorm. And the statement could be written

$$\exists x \in D, \exists y\Big(F(x, y) \wedge M(y)\Big) \Rightarrow \forall z \in D, Q(z)$$

Exercise 4. Analyze the logical form of the following statements:

(a) Jane saw a police officer, and Roger saw one too.

(b) Jane saw a police officer, and Roger saw him too.

(c) Nobody in calculus is smarter than everyone in discrete math.

(d) Anyone who has bought a Rolls Royce with cash must have a rich uncle.

(e) Every number that is larger than x is larger than y.

5 Partially ordered sets

Definition 9. Let P be a set, and let $\leq$ be a relation on S. $\leq$ is a partial order on the set P if:

(i) $\forall x \in P, x \leq x$ (reflexive)

(ii) $\forall x, y \in P, \Big(\big(x \leq y \wedge y \leq x \big) \Rightarrow x = y \Big)$ (antisymmetric)

(iii) $\forall x, y, z \in P, \Big(\big(x \leq y \wedge y \leq z \big) \Rightarrow x \leq z \Big)$ (transitive)

Property (i) in definition 9 says that every element in P is comparable to itself. Property (ii) says that if for any x, y in P, if the order relation holds in both directions, then $x = y$.

Example: The set of real numbers, $\mathbb{R}$, is ordered with the usual weak inequality, $\leq$.

More information and another example is given in the notes posted on the webpage.

6 Modular arithmetic

First we will formally describe the familiar notion of division of integers.

Lemma 1. *For any two integers, a and $b \neq 0$, there exists two unique integers, s and $r \geq 0$, such that $a = s \times b + r$ and $r < |b|$*

Example: For 13 and 3, we have that $11 = 4 \times 3 + 1$, so $s = 4$ and $r = 1$.

For an integer, n, $\mod n$ *partitions* the set of integers, or groups the integers into disjoints sets, based on the remainder when dividing by n. The disjoint statement can be seen since the remainder, r, in division is unique, that means each integer has one remainder, hence belongs in one set of the partition created by $\mod n$.

For example, $8 = 2 \times 3 + 2$, so $8 \mod 3$ is 2. Now what we really mean by $8 \mod 3$ is that 8 belongs to the set of all integers that have remainder 2 when dividing by 3. We say 8 is congruent to 2 mod 3, written $2 \equiv 2 \mod 3$.

If, for an integer, n, we have that two integers, a, b are congruent mod$n$, $a \equiv b \mod n$, we have $b \equiv a \mod n$. Also, if $a \equiv b \mod n$, then we have $(a - b) \equiv 0 \mod n$, which means that $a - b$ is a multiple of n. This can be seen by writing out $a \equiv b \mod n$ using division as follows: For some integer s, we have

$$b \equiv a \mod n \Rightarrow b = s \times n + a$$

$$\Rightarrow b - a = s \times n \Rightarrow b - a \equiv 0 \mod n$$

The mod relation is an example of a more general relation called an *equivalence relation*. An equivalence relation on a set partitions that set, and, conversely, any partition of a set defines an equivalence relation on that set.

Example: Let $\bar{r}$ =the set of integers congruent to $r \mod 3$. Each integer mod 3 is congruent to $0, 1$, or 2, so each integer $\mod 3$ can be represented by the set $\{\bar{0}, \bar{1}, \bar{2}\}$. And in general, each integer $\mod n$ can be represented by the set $\{\bar{0}, \bar{1}, \bar{2}, ..., \overline{n-1}\}$.