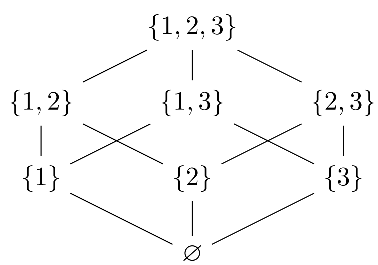
MGF1107 Notes

A partial order on a set is a relation that compares the elements of the set. The subset relation, $\subseteq$, is a partial order.

For example, take the set $\{1,2,3\}$, and let P be the set of all subsets of $\{1,2,3\}$. So P is a set whose elements are themselves sets.

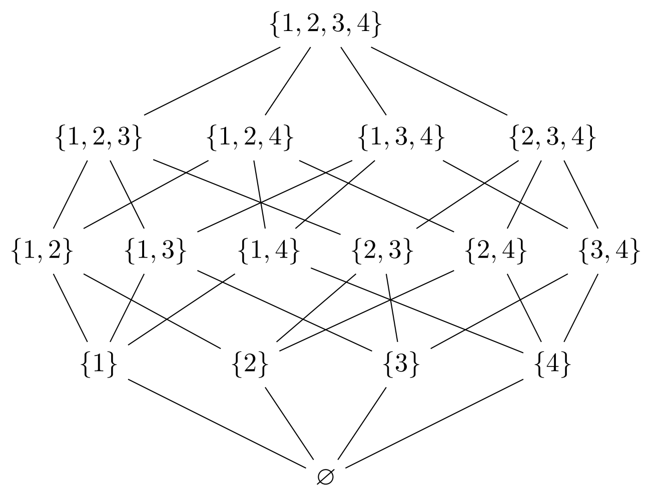
$$P = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

The subset relation partially orders P. The element of P, which is a set, $\{1\}$ is "less than" $\{1,2\}$ since $\{1\} \subseteq \{1,2\}$. The word 'partially' is used because there are some elements of P that cannot be compared by the subset relation. The element $\{1\} \nsubseteq \{2\}$ and $\{2\} \nsubseteq \{1\}$. We can use Hasse diagrams to illustrate the subset order relation as shown below.



The "larger" elements are at the top, and the lines connect to "smaller" elements below. The incomparable elements $\{1\}$ and $\{2\}$ have no line connecting them. Note that the elements of a single row are all incomparable.

The diagram for the set of subsets of $\{1,2,3,4\}$ is



Notice that for $\{1,2,3\}$, there are $8 = 2^3$ subsets, or elements, in P, and for $\{1,2,3,4\}$ there are $16 = 2^4$ elements. The structure of subsets of a set with n elements (with the subset order) is identical to binary strings of length n with a partial order, $\leq$. By binary string of length n, we mean a sequence of n zeros and ones. For example, 0001 and 1010 are binary strings of length 4. The comparison can be seen if we think of the ones in a string as representing which elements from $\{1, ..., n\}$ will be taken to form a subset. So 1100 could correspond to $\{1,2\}$ and 0101 could correspond to $\{2,4\}$. The corresponding order on binary strings, $\leq$, is that if each number in a string, x, is less than or equal to the corresponding number in another string, y, then $x \leq y$.

You may be wondering whether we can determine how many elements will be in a given row of the diagram. There is. Google binomial theorem.