

# Global stability of a DS-DI epidemic model with age of infection <sup>\*</sup>

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## Abstract

A model with differential susceptibility, differential infectivity (DS-DI), and age of infection is formulated in this paper. The susceptibles are divided into  $n$  groups according to their susceptibilities. The infectives are divided into  $m$  groups according to their infectivities. The total population size is assumed constant. Formula for the reproductive number is derived so that if the reproduction number is less than one, the infection-free equilibrium is locally stable, and unstable otherwise. Furthermore, if the reproductive number is less than one, the infection-free equilibrium is globally asymptotically stable. If the reproductive number is greater than one, it is shown that there exists a unique endemic equilibrium which is globally asymptotically stable. This result is obtained through a Lyapunov function.

**Keywords:** infection-age; reproductive number; global stability; Lyapunov function.

## 1 Introduction

Genetic variability of susceptible individuals may lead to their differentiation in susceptibility to infection. Genetic predisposition of some individuals to some illnesses is

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well documented in the literature. For instance, tuberculosis is a centuries old disease of humans. Development of genetic tools have now given evidence that different individuals exhibit different susceptibility to the disease [1]. More recently, avian influenza H5N1 strains have started to infect humans mostly through bird-to-human transmission. However, limited human-to-human transmission also occurs primarily in families, suggesting possible genetic predisposition to the disease. Genetic evidence now exists that individuals may be exhibiting differential susceptibility to the infection, as well as subsequent different severity of the disease [2]. Host genetic factors play a major role in determining the susceptibility to infectious diseases. Further studies are needed to determine the hosts' differential susceptibility to various disease as well as its implications to public health. Here we develop a model that takes into account the differential susceptibilities (DS) of individuals.

In the study of HIV transmission, acute primary infection is followed by a chronic phase. During the chronic phase, HIV RNA levels drop several orders of magnitude and remain "nearly constant" for years [3, 4], Viral levels differ by many orders of magnitude between individuals after the acute phase. People with high viral loads in the chronic phase tend to progress rapidly to AIDS, whereas those with very low loads tend to be slow or non-progressors [5, 6, 7]. To account for such differences between infected individuals, a differential infectivity (DI) SIR model was proposed in [8], where the infected population is subdivided into  $n$  subgroups,  $I_1, I_2, \dots, I_n$ . Differential infectivity is not unique only to HIV transmission. Viral levels, differing between individuals, have been shown in other diseases such as other sexually transmitted diseases, characterized by the so-called core groups [9, 10], as well as malaria, and dengue fever, where the infectivity depends on parasite or viral loads in infected hosts or vectors [11, 12].

In [13], James M. Hyman and Jia Li, formulated compartmental differential susceptibility (DS) susceptible-infective-removed (SIR) models by dividing the susceptible population into multiple subgroups according to the susceptibility of individuals in each group. They derived an explicit formula for the reproductive number of infection for each model. They further proved that the infection-free equilibrium and endemic equilibria of each model were globally asymptotically stable. In [14], Zhien Ma et al, presented several differential infectivity (DI) epidemic models under different assumptions. They established global stability of the infection-free equilibrium and the endemic equilibrium for DI models of SIR (susceptible/infected/removed) type with bilinear incidence and standard incidence but no disease-induced death. In [15], James M. Hyman, and Jia Li, formulated differential susceptibility and differential infectivity models for disease transmission. They obtained explicit formulas for the reproductive number. They showed that the infection-free equilibrium is globally stable and there exists a unique endemic equilibrium for these models. In contrast with the above mentioned models, in this paper, we formulate a system of partial differential equations and perform global stability of equilibria.

To gain insight into the transmission dynamics of diseases with differential susceptibility and differential infectivity, the susceptibility and the infectivity are coupled and can not be separated. Results on couples' studies for HIV transmission suggest that differences may be due to variability in both susceptibility and infectivity. To further understand these phenomena, in this paper we propose a combined differential susceptibility and differential infectivity (DS-DI) epidemic model with age of infection, where recovered infectives are either completely removed or isolated, or they have full immunity after they recovery. The dynamics of the model are governed by a mixed system of ordinary differential equations and partial differential equations.

## 2 The model formulation

We consider the spread of a disease in a randomly mixing population that approaches a steady state  $S^0$ , in the absence of infection. We assume that infected individuals who are removed from the population after they have recovered from infection, become fully immune. We approximate the transmission dynamics with an SIR (Susceptible  $\rightarrow$  Infective  $\rightarrow$  Recovered) model. We assume that susceptible individuals may have different susceptibilities and divide them into  $n$  groups,  $S^1, S^2, \dots, S^n$ . Hence, the individuals in each group have homogeneous susceptibility, but the susceptibilities of individuals from different groups are distinct. The susceptibles are distributed into  $n$  susceptible subgroups based on their inherent susceptibilities. This is done in such a way that the input flow into group  $S^k$  is  $p^k \mu S^0$  with  $\sum_{k=1}^n p^k = 1$ . The infectives are divided into  $m$  groups  $i_j(t, \tau)$ ,  $j = 1, \dots, m$ , which are assumed to be functions of age of infection  $\tau$ . Each infective group is assumed to have different infectivity progression as a function of the time since infection. A susceptible individual in group  $S^k$  enters an infectious group  $i_j(t, \tau)$  with probability  $q_{kj}$  and stays in this group until becoming recovered or removed. We must have  $\sum_{j=1}^m q_{kj} = 1$  for  $k = 1, \dots, n$ .

We assume full immunity of recovered individuals, or complete isolation after individuals are infected and diagnosed, and we group all these individuals in group  $R$ . The transmission dynamics of infection are governed by the following differential equations

$$\left\{ \begin{array}{l} \frac{dS^k}{dt} = \mu(p^k S^0 - S^k(t)) - \lambda_k(t)S^k(t), \quad k = 1, \dots, n, \\ \frac{\partial i_j(t, \tau)}{\partial t} + \frac{\partial i_j(t, \tau)}{\partial \tau} = -(\mu + v_j(\tau))i_j(t, \tau), \quad j = 1, \dots, m, \\ i_j(t, 0) = \sum_{k=1}^n q_{kj} \lambda_k(t) S^k(t), \\ \frac{dR}{dt} = \sum_{j=1}^m \int_0^\infty v_j(\tau) i_j(t, \tau) d\tau - \mu R(t), \end{array} \right. \quad (2.1)$$

where  $\mu$  is the natural death rate in the absence of infection, and  $v_j(\tau)$  is the age-since-infection dependent recovery or removal rate from group  $i_j(t, \tau)$ . The force of infection for the susceptibles in group  $S^k$  is given by

$$\lambda_k(t) = \frac{\alpha_k}{N} \sum_{l=1}^m \int_0^\infty \beta_l(\tau) i_l(t, \tau) d\tau$$

where  $N$  denotes the total population size. The total population size is given by:

$$N(t) = \sum_{k=1}^n S^k(t) + \sum_{j=1}^m \int_0^\infty i_j(t, \tau) d\tau + R(t).$$

Model (2.1) is equipped with the following initial conditions:

$$S^k(0) = S_{k0} \geq 0, \quad i_j(0, \tau) = i_{j0}(\tau) \in L_+^1(0, \infty), \quad R(0) = R_0 \geq 0.$$

The parameter  $\alpha_k$  gives the susceptibility of susceptible individuals in group  $S^k$ , and  $\beta_j(\tau)$  is the age-since-infection dependent infectivity in group  $i_j(t, \tau)$ . We denote by  $C_{B,U}([0, +\infty), \mathfrak{R})$  the set of bounded and uniformly continuous mappings from  $[0, +\infty)$  to  $\mathfrak{R}$ .

**Assumption 2.1.** Suppose that

- (a)  $\mu, \delta, S^0 \in (0, +\infty)$ ;
- (b)  $v_j(\tau) \in L_+^\infty(0, +\infty), j = 1, \dots, m$ ;
- (c)  $\beta_j(\tau) \in C_{B,C}[0, +\infty), \mathfrak{R}) \cap C_+([0, +\infty), \mathfrak{R})$ .

Summing the equations of (2.1), we have that the total population size  $N(t)$  satisfies the differential equation

$$N'(t) = \mu S^0 - \mu N(t),$$

whose solution is given by the formula

$$N(t) = N_0 e^{-\mu t} + S^0 (1 - e^{-\mu t}).$$

Therefore, we can assume the total population is constant, that is,  $N(t) = S^0$ . Throughout this article we will work with the normalized system of system (2.1). We introduce the following fractions:

$$s^k(t) = \frac{S^k(t)}{N}, \quad i^j(t, \tau) = \frac{i_j(t, \tau)}{N}, \quad k = 1, \dots, n, \quad j = 1, \dots, m, \quad r(t) = \frac{R(t)}{N}.$$

We note that the proportions do not exceed one for any value  $\tau$  and  $t$ . The normalized force of infection is given by

$$\lambda^k(t) = \alpha_k \sum_{l=1}^m \int_0^\infty \beta_l(\tau) i^l(t, \tau) d\tau.$$

We can rewrite system (2.1) as follows:

$$\begin{cases} \frac{ds^k}{dt} = \mu(p^k - s^k(t)) - \lambda^k(t)s^k(t), & k = 1, \dots, n, \\ \frac{\partial i^j(t, \tau)}{\partial t} + \frac{\partial i^j(t, \tau)}{\partial \tau} = -(\mu + v_j(\tau))i^j(t, \tau), & j = 1, \dots, m, \\ i^j(t, 0) = \sum_{k=1}^n q_{kj}\lambda^k(t)s^k(t), \\ \frac{dr}{dt} = \sum_{j=1}^m \int_0^\infty v_j(\tau)i^j(t, \tau)d\tau - \mu r(t). \end{cases} \quad (2.2)$$

System (2.2) has a unique disease-free equilibrium  $E_0 = x_f = (s_f^k, 0, 0)$ , and an endemic equilibrium  $E_* = (s_e^k, i_e^j(\tau), r_e)$ ,  $k = 1, \dots, n, j = 1, \dots, m$ .

Set  $X = \mathfrak{R}^n \times Y^m \times \mathfrak{R}$ , where  $Y = \mathfrak{R} \times L^1(0, +\infty)$ . Furthermore, we define

$$X_+ = \mathfrak{R}_+^n \times Y_+^m \times \mathfrak{R}_+, \quad X_0 = \mathfrak{R}^n \times Y_0^m \times \mathfrak{R}, \quad X_{+0} = X_0 \cap X_+$$

with

$$Y_+ = \mathfrak{R}_+ \times L_+^1(0, \infty), \quad Y_0 = \{0\} \times L^1(0, \infty).$$

Define the linear operator  $\mathcal{A}_j : D(\mathcal{A}_j) \subset Y \rightarrow Y$  as follows

$$\mathcal{A}_j \begin{pmatrix} 0 \\ \phi_j \end{pmatrix} = \begin{pmatrix} -\phi_j(0) \\ -\phi_j' - (\mu + v_j(\tau))\phi_j \end{pmatrix}.$$

If  $\lambda \in C$ , with  $\operatorname{Re} \lambda > -\mu$ , then  $\lambda \in \rho(\mathcal{A}_j)$ ,  $\rho(\mathcal{A}_j)$  represents the resolvent set of  $\mathcal{A}_j$ , and we have the following explicit formula for the resolvent of  $\mathcal{A}_j$ ,

$$(\lambda I - \mathcal{A}_j)^{-1} \begin{pmatrix} \theta \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ \phi_j \end{pmatrix},$$

then

$$\phi_j(\tau) = B_j(\tau)e^{-(\lambda+\mu)\tau}\theta + \int_0^\tau B_j(\tau-s)e^{-(\lambda+\mu)(\tau-s)}\psi(s)ds,$$

where  $B_j(\tau) = e^{-\int_0^\tau v_j(\xi)d\xi}$ ,  $j = 1, \dots, m$ . If we define by  $i^j(t) = \begin{pmatrix} 0 \\ i^j(t, \cdot) \end{pmatrix}$ , the PDE equation in (2.2) can be rewritten in the form of an ordinary differential equation as a non-densely defined Cauchy problem

$$\begin{cases} \frac{ds^k(t)}{dt} = -\mu s^k(t) + F^k(s^k(t), i^j(t)), & k = 1, 2, \dots, n, \\ \frac{d}{dt} \begin{pmatrix} 0 \\ i^j(t, \cdot) \end{pmatrix} = \mathcal{A}_j \begin{pmatrix} 0 \\ i^j(t, \cdot) \end{pmatrix} + F^j(s^k(t), i^j(t)), & j = 1, 2, \dots, m, \\ \frac{dr}{dt} = -\mu r(t) + F(i^j(t, \cdot)), \end{cases} \quad (2.3)$$

where

$$F^k(s^k(t), \begin{pmatrix} 0 \\ i^j(t, \cdot) \end{pmatrix}) = \mu p^k - \lambda^k(t) s^k(t),$$

$$F^j(s^k(t), \begin{pmatrix} 0 \\ i^j(t, \cdot) \end{pmatrix}) = \begin{pmatrix} \sum_{k=1}^n q_{kj} \lambda^k(t) s^k(t) \\ 0 \end{pmatrix},$$

and

$$F(i^j(t, \cdot)) = \sum_{j=1}^m \int_0^\infty v_j(\tau) i^j(t, \tau) d\tau.$$

Let  $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$  be the linear operator defined by

$$\begin{aligned} \mathcal{A} \begin{pmatrix} s^k(t) \\ \begin{pmatrix} 0 \\ i^j(t, \cdot) \end{pmatrix} \\ r(t) \end{pmatrix} &= \begin{pmatrix} -\mu s^k \\ \mathcal{A}_j \begin{pmatrix} 0 \\ i^j(t, \cdot) \end{pmatrix} \\ -\mu r(t) \end{pmatrix} \\ &= \begin{pmatrix} -\mu & 0 & 0 \\ 0 & \mathcal{A}_j & 0 \\ 0 & 0 & -\mu \end{pmatrix} \begin{pmatrix} s^k(t) \\ \begin{pmatrix} 0 \\ i^j(t, \cdot) \end{pmatrix} \\ r(t) \end{pmatrix}. \end{aligned}$$

Let  $D(\mathcal{A}) = \mathfrak{R}^n \times Z^m \times \mathfrak{R}$  with  $Z = \{0\}^m \times W^{1,1}(0, +\infty)$ , and then  $X_0 = \overline{D(\mathcal{A})}$ , and  $X_{0+} = \overline{D(\mathcal{A})} \cap X_+$ . Then  $\overline{D(\mathcal{A})} = X_0$  is not dense in  $X$ . We introduce the non-linear map  $\mathcal{F} : \overline{D(\mathcal{A})} \rightarrow X$  defined by

$$\mathcal{F} \begin{pmatrix} s^k(t) \\ \begin{pmatrix} 0 \\ i^j(t, \cdot) \end{pmatrix} \\ r(t) \end{pmatrix} = \begin{pmatrix} F^k(s^k(t), \begin{pmatrix} 0 \\ i^j(t, \cdot) \end{pmatrix}) \\ F^j(s^k(t), \begin{pmatrix} 0 \\ i^j(t, \cdot) \end{pmatrix}) \\ F(i^j(t, \cdot)) \end{pmatrix}.$$

With these definitions, we can rewrite system (2.2) as the following abstract Cauchy problem

$$\begin{aligned} \frac{du(t)}{dt} &= \mathcal{A}u(t) + \mathcal{F}(u(t)), t \geq 0, \\ u(0) &= x \in \overline{D(\mathcal{A})}. \end{aligned}$$

The fact that the nonlinearities are Lipschitz continuous on bounded sets leads to the following proposition.

**Lemma 2.2.**([16]) *There exists a uniquely determined semiflow  $\{U(t)\}_{t \geq 0}$  on  $X_{0+}$ ,*

*such that for each  $x = \begin{pmatrix} s_0^k \\ \begin{pmatrix} 0 \\ i_0^j \end{pmatrix} \\ r_0 \end{pmatrix} \in X_{0+}$ , there exists a unique continuous map*

$U \in C([0, +\infty), X_{0+})$  that gives an integrated solution of the Cauchy problem (2.2), that is,

$$\int_0^t U(s)x ds \in D(\mathcal{A}), \forall t \geq 0,$$

and

$$U(t)x = x + \mathcal{A} \int_0^t U(s)x ds + \int_0^t \mathcal{F}(U(s)x) ds, \forall t \geq 0.$$

To compute the projector on the eigenspace associated with the dominant eigenvalue, we now derive the linearized equations at the disease-free equilibrium. In addition, we study the uniform persistence of the system. The linearized equations at the disease-free equilibrium  $(s_f^k, 0_j, 0)$ ,  $k = 1, \dots, n, j = 1, \dots, m$  are given by:

$$\begin{cases} \frac{ds^k(t)}{dt} = -\mu s^k(t) - \alpha_k p^k \sum_{j=1}^m \int_0^\infty \beta_j(\tau) i_j(t, \tau) d\tau s_f^k, \\ \frac{\partial i^j(t, \tau)}{\partial t} + \frac{\partial i^j(t, \tau)}{\partial \tau} = -(\mu + v_j(\tau)) i^j(t, \tau), \\ i^j(t, 0) = \sum_{k=1}^n q_{kj} \alpha_k \sum_{l=1}^m \int_0^\infty \beta_l(\tau) i_l(t, \tau) d\tau s_f^k, \\ \frac{dr(t)}{dt} = \sum_{j=1}^m \int_0^\infty v_j(\tau) i_j(t, \tau) d\tau - \mu r(t), \\ s^k(0) = s_0^k, i^j(0, \cdot) = i_0^j \in L_+^1(0, +\infty), r(0) = r_0 \geq 0. \end{cases}$$

Next we study the spectral properties of the linearized equations.

**Definition 2.3.** ([17]) Let  $L : D(L) \subset X \rightarrow X$  be the infinitesimal generator of a linear  $C_0$ -semigroup  $T_L(t)_{t \geq 0}$  on a Banach space  $X$ . We define the growth bound  $w_0(L) \in [-\infty, +\infty)$  of  $L$  by

$$w_0(L) := \lim_{t \rightarrow +\infty} \frac{\ln(\|T_L(t)\|_X)}{t}.$$

The essential growth bound  $w_{0,ess}(L) \in [-\infty, +\infty)$  of  $L$  is defined by

$$w_{0,ess}(L) := \lim_{t \rightarrow +\infty} \frac{\ln(\|T_L(t)\|_{ess})}{t},$$

where  $\|T_L(t)\|_{ess}$  is the essential norm of  $T_L(t)$  defined by

$$\|T_L(t)\|_{ess} = \kappa(T_L(t)B_X(0, 1)).$$

Here  $B_X(0, 1) = \{x \in X : \|x\|_X \leq 1\}$ , and for each bounded set  $B \subset X$ ,

$$\kappa(B) = \inf\{\varepsilon > 0 : B \text{ can be covered by a finite number of balls of radius } \leq \varepsilon\}$$

is the Kuratovsky measure of non-compactness.

The following Theorem shows the existence of a projector. The existence of a projector was first proved by Webb [20, 21] and the fact that there is a finite number of points of the spectrum has been proved by Engel and Nagel [22]. We denote by  $L_Y : D(L_Y) \in Y \rightarrow Y$  the part of  $L$  on  $Y$ , which is defined by

$$L_Y x = Lx, \quad \forall x \in D(L_Y) := \{x \in D(L) \cap Y : Lx \in Y\}.$$

**Theorem 2.4.** *Let  $L : D(L) \subset X \rightarrow X$  be the infinitesimal generator of a linear  $C_0$ -semigroup  $\{T_L(t)\}$  on a Banach space  $X$ . Then*

$$w_0(L) = \max(w_{0,ess}(L), \max_{\lambda \in \sigma(L) \setminus \sigma_{ess}(L)} \operatorname{Re}(\lambda)).$$

Assume in addition that  $w_{0,ess}(L) < w_0(L)$ . Then for each  $\gamma \in (w_{0,ess}(L), w_0(L)]$  the set  $\{\lambda \in \sigma(L) : \operatorname{Re}(\lambda) \geq \gamma\} \subset \sigma_p(L)$  is nonempty, finite and contains only poles of the resolvent of  $L$ . Moreover, there exists a finite rank bounded linear projector  $\Pi : X \rightarrow X$  satisfying the following properties:

- (a)  $\Pi(\lambda - L)^{-1} = (\lambda - L)^{-1}\Pi, \forall \lambda \in \rho(L)$ ;
- (b)  $\sigma(L_{\Pi(X)}) = \{\lambda \in \sigma(L) : \operatorname{Re}(\lambda) \geq \gamma\}$ ;
- (c)  $\sigma(L_{(I-\Pi)(X)}) = \sigma(L) \setminus \sigma(L_{\Pi(X)})$ .

For the linearized system, the dynamics of  $i^j$  do not depend on  $s^k$  and  $r$ . In order to study the uniform persistence of the disease, we need to focus on the linear system

$$\begin{cases} \frac{\partial i^j(t, \tau)}{\partial t} + \frac{\partial i^j(t, \tau)}{\partial \tau} = -(\mu + v_j(\tau))i^j(t, \tau), \\ i^j(t, 0) = \sum_{k=1}^n q_{kj} \alpha_k \sum_{l=1}^m \int_0^\infty \beta_l(\tau) i^l(t, \tau) d\tau s_f^k, \\ i^j(0, \cdot) = i_0^j \in L_+^1(0, +\infty), \end{cases}$$

where  $s_f^k = p^k, k = 1, \dots, n$ . We define

$$\mathcal{B}_j \begin{pmatrix} 0 \\ \phi_j \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^n q_{kj} \alpha_k \sum_{l=1}^m \int_0^\infty \beta_l(\tau) i^l(t, \tau) d\tau s_f^k \\ 0 \end{pmatrix}.$$

For  $\lambda \in C$  with  $\operatorname{Re}(\lambda) > -\mu$ , we defined the characteristic function  $\Delta_j(\lambda)$  as

$$\Delta_j(\lambda) = 1 - \sum_{j=1}^m \sum_{k=1}^n q_{kj} p^k \alpha_k \int_0^\infty \beta_j(\tau) B_j(\tau) e^{-(\lambda + \mu)\tau} d\tau.$$

Then we have

$$\begin{aligned}
& (\lambda I - (\mathcal{A}_j + \mathcal{B}_j)) \begin{pmatrix} 0 \\ \phi_j \end{pmatrix} = \begin{pmatrix} \theta \\ \psi \end{pmatrix} \\
& \Leftrightarrow (\lambda I - \mathcal{A}_j) \begin{pmatrix} 0 \\ \phi_j \end{pmatrix} = \begin{pmatrix} \theta \\ \psi \end{pmatrix} + \mathcal{B}_j \begin{pmatrix} 0 \\ \phi_j \end{pmatrix} \\
& \Leftrightarrow \phi_j(\tau) = B_j(\tau) e^{-(\lambda+\mu)\tau} \theta + \int_0^\tau B_j(\tau-s) e^{-(\lambda+\mu)(\tau-s)} \psi(s) ds \\
& \quad + B_j(\tau) e^{-(\lambda+\mu)\tau} \sum_{k=1}^n q_{kj} p^k \alpha_k \sum_{l=1}^m \int_0^\infty \beta_l(\tau) \phi_l(\tau) d\tau.
\end{aligned}$$

Thus,

$$\begin{aligned}
& \Delta_j(\lambda) \sum_{k=1}^n q_{kj} p^k \alpha_k \sum_{l=1}^m \int_0^\infty \beta_l(s) \phi_l(s) d\tau \\
& = \sum_{k=1}^n q_{kj} p^k \alpha_k \sum_{l=1}^m \int_0^\infty \beta_l(s) B_l(s) e^{-(\lambda+\mu)s} ds \theta \\
& \quad + \sum_{k=1}^n q_{kj} p^k \alpha_k \sum_{l=1}^m \int_0^\infty \beta_l(\tau) \int_0^\tau B_l(\tau-s) e^{-(\lambda+\mu)(\tau-s)} \psi(s) ds d\tau,
\end{aligned}$$

so

$$\begin{aligned}
\phi_j(\tau) = & B_j(\tau) e^{-(\lambda+\mu)\tau} [1 + \Delta_j(\lambda)^{-1} \sum_{k=1}^n q_{kj} p^k \alpha_k \sum_{l=1}^m \int_0^\infty \beta_l(s) B_l(s) e^{-(\lambda+\mu)s} ds] \theta \\
& + \Delta_j(\lambda)^{-1} \sum_{k=1}^n q_{kj} p^k \alpha_k \sum_{l=1}^m \int_0^\infty \beta_l(\tau) \int_0^\tau B_l(\tau-s) e^{-(\lambda+\mu)(\tau-s)} \psi(s) ds d\tau \\
& + \int_0^\tau B_j(\tau) e^{-(\lambda+\mu)(\tau-s)} \psi(s) ds,
\end{aligned}$$

we note that

$$1 + \Delta_j(\lambda)^{-1} \sum_{k=1}^n q_{kj} p^k \alpha_k \sum_{l=1}^m \int_0^\infty \beta_l(\tau) B_l(\tau-s) e^{-(\lambda+\mu)s} ds = \Delta_j(\lambda)^{-1}.$$

Then, we have:

$$\begin{aligned}
\phi_j(\tau) = & B_j(\tau) e^{-\int_0^\tau (\lambda+\mu) ds} \{ \Delta_j(\lambda)^{-1} [ \sum_{k=1}^n q_{kj} p^k \alpha_k \sum_{l=1}^m \int_0^\infty \beta_l(\tau) \int_0^\tau B_l(\tau-s) e^{-\int_s^\tau (\lambda+\mu) d\sigma} \\
& \psi(s) ds d\tau + \theta ] \} + \int_0^\tau B_j(\tau-s) e^{-(\lambda+\mu)(\tau-s)} \psi(s) ds.
\end{aligned}$$

Define the *reproduction number*

$$R_0 = \sum_{j=1}^m \sum_{k=1}^n q_{kj} p^k \alpha_k \int_0^\infty \beta_j(\tau) B_j(\tau) e^{-\mu\tau} d\tau.$$

Assume that the reproduction number satisfies  $R_0 > 1$ . Then we can find  $\lambda_0 \in \Re$ , such that

$$\sum_{j=1}^m \sum_{k=1}^n q_{kj} p^k \alpha_k \int_0^\infty \beta_j(\tau) B_j(\tau) e^{-(\lambda_0+\mu)\tau} d\tau = 1.$$

Furthermore,  $\lambda_0 > 0$  is the dominant eigenvalue of  $\mathcal{A}_j + \mathcal{B}_j$  (see[20]). Moreover, we have

$$-\frac{d\Delta_j(\lambda_0)}{d\lambda} = \sum_{j=1}^m \sum_{k=1}^n q_{kj} p^k \alpha_k \int_0^\infty \tau \beta_j(\tau) B_j(\tau) e^{-\int_0^\tau (\lambda_0 + \mu) d\sigma} d\tau > 0.$$

Noticing that

$$\Pi_j \begin{pmatrix} \theta \\ \psi \end{pmatrix} = \lim_{\lambda \rightarrow \lambda_0} (\lambda - \lambda_0) (\lambda I - (\mathcal{A}_j + \mathcal{B}_j))^{-1} \begin{pmatrix} \theta \\ \psi \end{pmatrix},$$

and it satisfies

$$\Pi_j \begin{pmatrix} \theta \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ \phi_j \end{pmatrix}.$$

Thus, we have

$$\begin{aligned} \phi_j(\tau) = & B_j(\tau) e^{-(\lambda_0 + \mu)\tau} \left\{ \left( \frac{d\Delta(\lambda_0)}{d\lambda} \right)^{-1} \left[ \sum_{k=1}^n q_{kj} p^k \alpha_k \sum_{l=1}^m \int_0^\infty \beta_l(\tau) \int_0^\tau B_l(\tau - s) e^{-\int_s^\tau (\lambda + \mu) d\sigma} \psi(s) ds d\tau + \theta \right] \right\}. \end{aligned}$$

The linear operator  $\Pi_j : Y \rightarrow Y$  is the projector onto the generalized eigenspace of  $\mathcal{A}_j + \mathcal{B}_j$ , associated with the eigenvalue  $\lambda_0$ . We define  $\Pi : X \rightarrow X$

$$\Pi \begin{pmatrix} s^k \\ 0 \\ i^j \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ \Pi_j \begin{pmatrix} 0 \\ i^j \end{pmatrix} \\ 0 \end{pmatrix}.$$

We observe that the subset  $M_0$  can be defined by

$$M_0 = \{x \in X_{0+} : \Pi x \neq 0\}$$

and

$$\partial M_0 = X_{0+} \setminus M_0.$$

### Volterra Formulation:

Using Laplace transform arguments, one can establish that the mild solution of (2.2) takes the following form

$$U(t)x = \begin{pmatrix} s^k(t) \\ \begin{pmatrix} 0 \\ i^j(t, \cdot) \end{pmatrix} \\ r(t) \end{pmatrix},$$

where  $i^j(t, \tau)$  satisfies the following Volterra formulation of (2.2)

$$\begin{aligned} i^j(t, \tau) &= \begin{cases} i^j(t - \tau, 0)e^{-\mu\tau}B_j(\tau), & t \geq \tau, \\ i^j(0, \tau - t)e^{-\mu t}\frac{B_j(\tau)}{B_j(\tau - t)}, & t < \tau, j = 1, \dots, m, \end{cases} \\ &= \begin{cases} b^j(t - \tau)e^{-\mu\tau}B_j(\tau), & t \geq \tau, \\ i_{j0}(\tau - t)e^{-\mu t}\frac{B_j(\tau)}{B_j(\tau - t)}, & t < \tau, j = 1, \dots, m. \end{cases} \end{aligned} \quad (2.4)$$

The map  $b^j(\cdot) \in C([0, +\infty), \mathbb{R})$  is the unique solution of the following Volterra integral equation

$$\begin{aligned} b^j(t) &= \sum_{k=1}^n q_{kj}\alpha_k \sum_{l=1}^m \left( \int_0^t \beta_l(\tau)b^l(t - \tau)e^{-\mu\tau}B_l(\tau)d\tau \right. \\ &\quad \left. + \int_t^\infty \beta_l(\tau)i_{l0}(\tau - t)e^{-\mu t}\frac{B_l(\tau)}{B_l(\tau - t)}d\tau \right) s^k(t). \end{aligned} \quad (2.5)$$

We substitute (2.4) and (2.5) into the equations for  $s^k(t)$  and  $r(t)$ . According to [18], we can discuss the limiting system associated with (2.2),  $k = 1, \dots, n, j = 1, \dots, m$ ,

$$\begin{cases} \frac{ds^k}{dt} = \mu(p^k - s^k) - \alpha_k \sum_{l=1}^m \int_0^\infty \beta_l(\tau)b^l(t - \tau)B_l(\tau)e^{-\mu\tau}d\tau s^k(t), \\ b^j(t) = \sum_{k=1}^n q_{kj}\alpha_k \sum_{l=1}^m \int_0^\infty \beta_l(\tau)b^l(t - \tau)e^{-\mu\tau}B_l(\tau)d\tau s^k(t), \\ \frac{dr(t)}{dt} = \sum_{j=1}^m \int_0^\infty v_j(\tau)b^j(t - \tau)e^{-\mu\tau}B_j(\tau)d\tau - \mu r(t). \end{cases} \quad (2.6)$$

### 3 Stability of the infection-free equilibrium

In this section, we investigate the stability of the disease-free equilibrium. The stability of the disease-free equilibrium depends on the reproduction number  $R_0$ . We show that the infection-free equilibrium is globally stable as long as  $R_0 < 1$ , and it is unstable if  $R_0 > 1$ . The process takes two steps. In the first step we prove the local stability of the infection-free equilibrium as well as its instability. In the second step we prove that the disease-free equilibrium is globally asymptotically stable, that is, we show that system (2.2) has a global attractor in  $\partial M_0$ .

System (2.6) always has the disease-free equilibrium

$$E_0 = (s_f^1, \dots, s_f^n, 0, \dots, 0) = (p^1, \dots, p^n, 0, \dots, 0).$$

We linearize system (2.6) about  $E_0$  by defining the perturbation variables  $s^k(t) =$

$x^k(t) + s_f^k$ ,  $y^j(t) = b^j(t)$ ,  $k = 1, \dots, n$ ,  $j = 1, \dots, m$ ,  $r(t) = z(t)$ . We obtain the system

$$\begin{cases} \frac{dx^k(t)}{dt} = -\mu x^k(t) - \alpha_k p^k \sum_{l=1}^m \int_0^\infty \beta_l(\tau) y^l(t - \tau) B_j(\tau) e^{-\mu\tau} d\tau, \\ b^j(t) = \sum_{k=1}^n q_{kj} \alpha_k p^k \sum_{l=1}^m \int_0^\infty \beta_l(\tau) y^l(t - \tau) e^{-\mu\tau} B_l(\tau) d\tau, \\ \frac{dz}{dt} = \sum_{j=1}^m \int_0^\infty v_j(\tau) y^j(t - \tau) B_j(\tau) e^{-\mu\tau} d\tau - \mu z. \end{cases} \quad (3.1)$$

Let

$$x^k = x_{k0} e^{\lambda t}, \quad y_j(t) = y_{j0} e^{\lambda t}, \quad z = z_0 e^{\lambda t}, \quad k = 1, \dots, n, \quad j = 1, \dots, m,$$

where  $x_{k0}$ ,  $y_{j0}$ , and  $z_0$  are to be determined. Substituting  $x^k, y_j$  and  $z$  into (3.1), we obtain the following equations

$$\begin{cases} (\lambda + \mu) x_{k0} + \alpha_k p^k \sum_{j=1}^m \int_0^\infty \beta_j(\tau) y_{j0} e^{-\lambda\tau} B_j(\tau) e^{-\mu\tau} d\tau = 0, \quad k = 1, \dots, n, \\ y_{j0} - \sum_{k=1}^n q_{kj} \alpha_k p^k \sum_{l=1}^m \int_0^\infty \beta_l(\tau) y_{l0} e^{-\lambda\tau} e^{-\mu\tau} B_l(\tau) d\tau = 0, \quad j = 1, \dots, m, \\ - \sum_{j=1}^m \int_0^\infty v_j(\tau) y_{j0} e^{-\lambda\tau} e^{-\mu\tau} B_j(\tau) d\tau + (\lambda + \mu) z_0 = 0. \end{cases}$$

**Theorem 3.1.** *The infection-free equilibrium  $E_0$  is locally asymptotically stable if  $R_0 < 1$ , and it is unstable if  $R_0 > 1$ .*

**Proof.** Considering the linearization of system (2.6) at the point  $E_0$ , that is system (3.1), we let

$$\begin{aligned} \hat{K}_j(\lambda) &= \int_0^\infty \beta_j(\tau) e^{-\lambda\tau} B_j(\tau) e^{-\mu\tau} d\tau, \quad j = 1, \dots, m, \\ \hat{K}_j^1(\lambda) &= \int_0^\infty v_j(\tau) e^{-\lambda\tau} B_j(\tau) e^{-\mu\tau} d\tau, \quad j = 1, \dots, m. \end{aligned}$$

We get the following characteristic equation

$$\begin{vmatrix} D_1 & D_2 & 0 \\ D_3 & D_4 & 0 \\ 0 & D_5 & \lambda + \mu \end{vmatrix} = 0,$$

where

$$D_1 = \begin{pmatrix} \lambda + \mu & 0 & \dots & 0 \\ 0 & \lambda + \mu & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda + \mu \end{pmatrix}_{n \times n},$$

and

$$D_2 = \begin{pmatrix} \alpha_1 p_1 \hat{K}_1(\lambda) & \alpha_1 p_1 \hat{K}_2(\lambda) & \cdots & \alpha_1 p_1 \hat{K}_m(\lambda) \\ \alpha_2 p_2 \hat{K}_1(\lambda) & \alpha_2 p_2 \hat{K}_2(\lambda) & \cdots & \alpha_2 p_2 \hat{K}_m(\lambda) \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_m p_m \hat{K}_1(\lambda) & \alpha_m p_m \hat{K}_2(\lambda) & \cdots & \alpha_m p_m \hat{K}_m(\lambda) \end{pmatrix}_{n \times m}.$$

$D_3$  is a  $n \times m$  matrix, whose components are all 0.  $D_5 = (-\hat{K}_1^1(\lambda), -\hat{K}_2^1(\lambda), \dots, -\hat{K}_m^1(\lambda))$ .

$$D_4 = \begin{pmatrix} 1 - \sum_{k=1}^n q_{k1} \alpha_k p^k \hat{K}_1(\lambda) & - \sum_{k=1}^n q_{k1} \alpha_k p^k \hat{K}_2(\lambda) & \cdots & - \sum_{k=1}^n q_{k1} \alpha_k p^k \hat{K}_m(\lambda) \\ - \sum_{k=1}^n q_{k2} \alpha_k p^k \hat{K}_1(\lambda) & 1 - \sum_{k=1}^n q_{k2} \alpha_k p^k \hat{K}_2(\lambda) & \cdots & - \sum_{k=1}^n q_{k2} \alpha_k p^k \hat{K}_m(\lambda) \\ \cdots & \cdots & \cdots & \cdots \\ - \sum_{k=1}^n q_{km} \alpha_k p^k \hat{K}_1(\lambda) & - \sum_{k=1}^n q_{km} \alpha_k p^k \hat{K}_2(\lambda) & \cdots & 1 - \sum_{k=1}^n q_{km} \alpha_k p^k \hat{K}_m(\lambda) \end{pmatrix}_{m \times m} = 0.$$

Define  $L_j = \sum_{k=1}^n q_{kj} \alpha_k p^k$ ,  $j = 1, \dots, m$ , and  $V = (L_1, L_2, \dots, L_m)^T$ . Using mathematical induction, we can show that

$$D_4 V = \left( 1 - \sum_{j=1}^m \sum_{k=1}^n q_{kj} \alpha_k p^k \hat{K}_j(\lambda) \right) V.$$

Since  $L_j > 0$ ,  $j = 1, \dots, m$ , then it follows from M-matrix theory that each eigenvalue of  $|D_4|$  is determined by

$$\sum_{j=1}^m \sum_{k=1}^n q_{kj} \alpha_k p^k \hat{K}_j(\lambda) = 1. \quad (3.2)$$

Notice that  $\hat{K}_j(\lambda)$  is a decreasing function of  $\lambda$  where  $\lambda$  is real. Moreover,

$$\lim_{\lambda \rightarrow -\infty} \hat{K}_j(\lambda) = +\infty,$$

and

$$\lim_{\lambda \rightarrow +\infty} \hat{K}_j(\lambda) = 0.$$

Furthermore, the reproductive number is given by:

$$R_0 = \sum_{j=1}^m \sum_{k=1}^n q_{kj} \alpha_k p^k \int_0^\infty \beta_j(\tau) e^{-\mu\tau} B_j(\tau) d\tau.$$

Then (3.2) has a unique real solution  $\lambda^*$ , which is a positive real number, if  $R_0 > 1$ .

Let  $u = a + ib$  be a complex number, where  $i = \sqrt{-1}$ , which is a solution of the characteristic equation (3.2). Then by separating the real and the imaginary part of  $\sum_{j=1}^m \sum_{k=1}^n q_{kj} \alpha_k p^k \hat{K}_j(u) = 1$  we obtain that the real part satisfies

$$1 = \sum_{j=1}^m \sum_{k=1}^n q_{kj} \alpha_k p^k \int_0^\infty \beta_j(\tau) e^{-\mu\tau} B_j(\tau) e^{-a\tau} \cos(b\tau) d\tau. \quad (3.3)$$

If  $\Re u = a \geq 0$ , then

$$\begin{aligned} 1 &= \sum_{j=1}^m \sum_{k=1}^n q_{kj} \alpha_k p^k \int_0^\infty \beta_j(\tau) e^{-\mu\tau} B_j(\tau) e^{-a\tau} \cos(b\tau) d\tau \\ &\leq \sum_{j=1}^m \sum_{k=1}^n q_{kj} \alpha_k p^k \int_0^\infty \beta_j(\tau) e^{-\mu\tau} B_j(\tau) d\tau = R_0 < 1. \end{aligned}$$

Hence, all solutions of (3.2) have negative real parts, if  $R_0 < 1$ . Therefore, the disease-free equilibrium is locally asymptotically stable if  $R_0 < 1$ , and unstable otherwise. In addition, we can obtain the following theorem through straightforward analysis.

**Theorem 3.2.** *If  $R_0 < 1$ , the infection-free equilibrium  $E_0$  is the unique attractor in  $\partial M_0$ .*

**Proof:** From the first equation of (2.6), it follows that

$$\frac{ds^k(t)}{dt} \leq \mu(p^k - s^k(t)),$$

that is

$$\begin{aligned} s^k(t) &\leq s_{k0} e^{-\mu t} + \int_0^t \mu p^k e^{-\mu(t-\tau)} d\tau \\ &= s_{k0} e^{-\mu t} + p^k - p^k e^{-\mu t}. \end{aligned}$$

Therefore, considering the limsup as  $t \rightarrow \infty$

$$\limsup_{t \rightarrow \infty} s^k(t) \leq p^k, k = 1, \dots, n. \quad (3.4)$$

Notice that

$$b^j(t) \leq \sum_{k=1}^n q_{kj} \alpha_k p^k \sum_{j=1}^m \int_0^\infty \beta_j(\tau) b^j(t - \tau) e^{-\mu\tau} B_j(\tau) d\tau.$$

Hence,

$$\limsup_{t \rightarrow \infty} b^j(t) \leq \sum_{k=1}^n q_{kj} \alpha_k p^k \sum_{l=1}^m \int_0^\infty \beta_l(\tau) e^{-\mu\tau} B_l(\tau) d\tau \limsup_{t \rightarrow \infty} b^j(t).$$

We multiply both sides by  $\int_0^\infty \beta_j(\tau) e^{-\mu\tau} B_j(\tau) d\tau$  and sum to obtain:

$$\begin{aligned} &\sum_{j=1}^m \int_0^\infty \beta_j(\tau) e^{-\mu\tau} B_j(\tau) d\tau \limsup_{t \rightarrow \infty} b^j(t) \\ &\leq \left( \sum_{j=1}^m \sum_{k=1}^n q_{kj} \alpha_k p^k \int_0^\infty \beta_j(\tau) e^{-\mu\tau} B_j(\tau) d\tau \right) \sum_{l=1}^m \int_0^\infty \beta_l(\tau) e^{-\mu\tau} B_l(\tau) d\tau \limsup_{t \rightarrow \infty} b^j(t). \end{aligned} \quad (3.5)$$

The coefficient on the right hand side of this inequality is exactly  $R_0$ . Since we assume  $R_0$  is strictly smaller than one, that is,  $R_0 < 1$ , the only way inequality (3.5) can hold is if  $\limsup_{t \rightarrow \infty} b^j(t) = 0, j = 1, \dots, m$ . From

$$\frac{dr(t)}{dt} = \sum_{j=1}^m \int_0^\infty v_j(\tau) b^j(t - \tau) e^{-\mu\tau} B_j(\tau) d\tau - \mu r(t),$$

it is easy to show that  $\limsup_{t \rightarrow \infty} r(t) = 0$ . Hence,  $\lim_{t \rightarrow \infty} s^k(t) = p^k, k = 1, \dots, n$ . This completes the proof of the theorem.

## 4 Existence and stability of the endemic equilibrium

In section 3 we have shown that if  $R_0 > 1$ , the infection-free equilibrium is unstable, and then the disease spreads when a small infection is introduced into the population. Now we assume  $R_0 > 1$ , and show that there exists an endemic equilibrium  $E_*$  whose components are positive. Furthermore, we show that this endemic equilibrium is globally stable.

For system (2.6), an endemic equilibrium needs to satisfy the equations

$$\begin{cases} \mu(p^k - s^{k*}) - \lambda^{k*} s^{k*} = 0, k = 1, \dots, n, \\ b^{j*} = \sum_{k=1}^n q_{kj} \alpha_k \sum_{l=1}^m \int_0^\infty \beta_l(\tau) b^{l*} e^{-\mu\tau} B_l(\tau) d\tau s^{k*}, j = 1, \dots, m, \\ \sum_{j=1}^m \int_0^\infty v_j(\tau) b^{j*} e^{-\mu\tau} B_j(\tau) d\tau - \mu r^* = 0. \end{cases} \quad (4.1)$$

We set

$$W := \sum_{j=1}^m \int_0^\infty \beta_j(\tau) b^{j*} e^{-\mu\tau} B_j(\tau) d\tau. \quad (4.2)$$

Solving (4.1) for  $s^{k*}$  and then for  $b^{j*}$  and  $r^*$  yields

$$\begin{aligned} s^{k*} &= \frac{\mu p^k}{\mu + \alpha_k W}, \quad k = 1, \dots, n, \\ b^{j*} &= \sum_{k=1}^n \alpha_k q_{kj} \frac{W \mu p^k}{\mu + \alpha_k W}, \quad j = 1, \dots, m, \\ r^* &= \frac{1}{\mu} \sum_{j=1}^m \int_0^\infty v_j(\tau) \sum_{k=1}^n \alpha_k q_{kj} \frac{\mu p^k W}{\mu + \alpha_k W} e^{-\mu\tau} B_j(\tau) d\tau, \end{aligned} \quad (4.3)$$

Substituting (4.3) into (4.2), we have

$$1 = \sum_{k=1}^n \sum_{j=1}^m \alpha_k q_{kj} p^k \int_0^{+\infty} \beta_j(\tau) B_j(\tau) e^{-\mu\tau} \frac{\mu}{\mu + \alpha_k W} d\tau. \quad (4.4)$$

Define a function  $H(W)$  by

$$H(W) = \sum_{k=1}^n \sum_{j=1}^m \alpha_k q_{kj} p^k \int_0^{+\infty} \beta_j(\tau) B_j(\tau) e^{-\mu\tau} \frac{\mu}{\mu + \alpha_k W} d\tau.$$

Notice that,

$$\lim_{W \rightarrow +\infty} H(W) = 0, \quad \lim_{W \rightarrow -\mu/\alpha} H(W) = +\infty,$$

where  $\alpha = \max\{\alpha_1, \dots, \alpha_n\}$ . Furthermore,  $H'(W) < 0$ .

Moreover, since  $H(0) = R_0 > 1$ , (4.4) has a unique positive solution  $W^*$ . Substituting  $W^*$  into (4.3), we have that system (2.6) has a unique positive endemic equilibrium  $E_*$ . We summarize this result in the following theorem.

**Theorem 4.1.** *There exists a unique positive equilibrium  $E_*$  if  $R_0 > 1$ .*

In the remainder of this section we assume that  $R_0 > 1$ , and we discuss the persistence of the system and the global stability of the endemic equilibrium  $E_*$ .

**Lemma 4.2.** ([19]) *Let  $X$  be locally compact, let  $X_2$  be compact in  $X$  and  $X_1$  be forward invariant under the continuous semiflow  $\Phi$  on  $X$ . Assume that*

$$\Omega_2 = \cup_{Y \in Y_2} w(y), \quad Y_2 = \{x \in X_2; \Phi_t(x) \in X_2, \forall t \geq 0\}$$

*has an acyclic isolated covering  $M = \cup_{k=1}^m M_k$ . If each part  $M_k$  of  $M$  is a weak repeller for  $X_1$ , then  $X_2$  is a uniform strong repeller for  $X_1$ .*

**Proposition 4.3.** *Assume that  $R_0 > 1$ . Then  $E_0 = x_f$  is ejective in  $M_0$  for  $\{U(t)\}_{t \geq 0}$ , that is  $M_0$  is uniform strong repeller for  $\partial M_0$ .*

**Proof.** We recall that  $R_0 > 1$ . Let  $\delta > 0$  and  $\epsilon_n \in (0, p^k)$  with  $\epsilon_n \in \mathfrak{R}_+$  satisfy  $\epsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ . We argue by contradiction. Assume that for each  $n \geq 0$ , we can

$$\text{find } x_n = \begin{pmatrix} s_0^{kn} \\ 0 \\ i_0^{jn} \\ r_0^n \end{pmatrix} \in \{y \in M_0 : \|x_f - y\| \leq \epsilon_n\}, \text{ such that}$$

$$\|x_f - U(t)x_n\| \leq \epsilon_n, \quad \forall t \geq 0. \quad (4.5)$$

Set

$$\begin{pmatrix} s^{kn}(t) \\ 0 \\ i^{jn}(t, \cdot) \\ r^n(t) \end{pmatrix} := U(t)x_n,$$

and we have

$$|s^{kn}(t) - s_f^k| \leq \epsilon_n, \quad \forall t \geq 0.$$

Moreover, if we denote by  $u^j(t) = i^j(t, 0)$ , for all  $t \geq 0$ , then

$$u^{jn}(t) = F_{jn}(t) + \sum_{k=1}^n q_{kj} \alpha_k s^{nk}(t) \sum_{l=1}^m \int_0^t \beta_l(\tau) B_l(\tau) e^{-\mu\tau} u^{ln}(t - \tau) d\tau, \quad \forall t \geq 0$$

with

$$F_{jn}(t) = \sum_{k=1}^n q_{kj} \alpha_k s^{nk}(t) \sum_{l=1}^m \int_t^{+\infty} \beta_l(\tau) i_0^{ln}(\tau) \frac{B_l(\tau)}{B_l(t - \tau)} e^{-\mu\tau} d\tau, \quad \forall t \geq 0.$$

Thus, the following inequality holds:

$$u^{jn}(t) \geq \sum_{k=1}^n q_{kj} \alpha_k (s_f^k - \epsilon_n) \sum_{l=1}^m \int_0^t \beta_l(\tau) B_l(\tau) e^{-\mu\tau} u^{ln}(t - \tau) d\tau. \quad (4.6)$$

Applying the Laplace transform to both sides of that inequality, we obtain

$$\hat{u}^{jn}(\lambda) \geq \sum_{k=1}^n q_{kj} \alpha_k(s_f^k - \epsilon_n) \sum_{l=1}^m \int_0^\infty \beta_l(\tau) B_l(\tau) e^{-(\mu+\lambda)\tau} d\tau \hat{u}^{ln}(\lambda)$$

where  $\hat{u}$  denotes the Laplace transform of  $u$ , and  $\lambda > 0$ . Multiplying both sides by  $\int_0^\infty \beta_j(\tau) B_j(\tau) e^{-(\mu+\lambda)\tau} d\tau$  and summing by  $j$ , we obtain the following inequality:

$$\begin{aligned} & \sum_{j=1}^m \int_0^\infty \beta_j(\tau) B_j(\tau) e^{-(\mu+\lambda)\tau} d\tau \hat{u}^{jn}(\lambda) \\ & \geq \left( \sum_{j=1}^m \sum_{k=1}^n q_{kj} \alpha_k(s_f^k - \epsilon_n) \int_0^\infty \beta_j(\tau) B_j(\tau) e^{-(\mu+\lambda)\tau} d\tau \right) \\ & \quad \sum_{l=1}^m \int_0^\infty \beta_l(\tau) B_l(\tau) e^{-(\mu+\lambda)\tau} d\tau \hat{u}^{ln}(\lambda) \end{aligned}$$

For  $\epsilon_n = 0$  and  $\lambda = 0$ , the quantity in the parentheses is exactly the reproduction number  $R_0$ . Since  $R_0 > 1$ , for  $\epsilon_n > 0$  and  $\lambda > 0$  but both small enough, this quantity is still larger than one, that is

$$\sum_{j=1}^m \sum_{k=1}^n q_{kj} \alpha_k(s_f^k - \epsilon_n) \int_0^\infty \beta_j(\tau) B_j(\tau) e^{-(\mu+\lambda)\tau} d\tau > 1$$

which is a contradiction.

**Proposition 4.4.** *Assume that  $R_0 > 1$ . The semiflow  $U(t)_{t \geq 0}$  is uniformly persistent with respect to the pair  $(\partial M_0, M_0)$ . More specifically, there exists  $\varepsilon > 0$ , such that*

$$\liminf_{t \rightarrow +\infty} \|\Pi U(t)x\| \geq \varepsilon.$$

*Moreover, there exists  $A_0$ , a compact subset of  $M_0$ , which is a global attractor for  $U(t)_{t \geq 0}$  in  $M_0$ .*

**Proof.** Since the disease free equilibrium  $E_0$  is globally asymptotically stable in  $\partial M_0$ , then the solutions of system (2.2) that start in  $\partial M_0$  are far from  $E_0$  as  $t \rightarrow -\infty$ . In addition,  $E_0 = x_f$  is ejective in  $M_0$  for  $\{U(t)\}_{t \geq 0}$  when  $R_0 > 1$ . The result on persistence follows from Lemma 4.2 and Proposition 4.3. This completes the proof of persistence.

In what follows, we discuss the global stability of the endemic equilibrium in  $A_0$ . Firstly, by using the Volterra's formulation of system (2.2), we obtain that

$$i^j(t, \tau) = b^j(t - \tau) e^{-\int_0^\tau (\mu + v_j(a)) da},$$

where

$$b^j(t) = \sum_{k=1}^n q_{kj} \lambda^k(t) s^k(t), j = 1, \dots, m.$$

From Proposition 4.4, we get the following lemma

**Lemma 4.5.** *There exist constants  $H > \varepsilon > 0$ , such that for every complete orbit of system (2.2) in an invariant set, we have*

$$\varepsilon \leq s^k(t) \leq H, \quad \forall t \in \mathbb{R},$$

$$\varepsilon \leq r(t) \leq H, \quad \forall t \in \mathbb{R},$$

and

$$\varepsilon \leq \sum_{j=1}^m \int_0^\infty \beta_j(\tau) i_j(t, \tau) d\tau \leq H, \quad \forall t \in \mathbb{R}.$$

Moreover,

$$O = \overline{\bigcup_{t \in \mathbb{R}} \{s^k(t), i^j(t, \cdot), r(t)\}}, k, = 1, \dots, n, j = 1, \dots, m$$

is compact in  $\mathbb{R}^n \times Y^m \times \mathbb{R}$ .

Let  $f(x) = x - 1 - \ln x$ . Note that  $f'(x) = 1 - \frac{1}{x}$ . Thus,  $f$  is decreasing on  $(0, 1]$  and increasing on  $[1, \infty)$ . The function  $f$  has only one extremum which is a global minimum at  $x = 1$ . The global minimum satisfies  $f(1) = 0$ . We first define expressions  $V_s^k(t)$ , and  $V_i^k(t)$ , and calculate their derivatives. Then, we analyze the Lyapunov function  $V = V_s^k + V_i^k$ . Let

$$V_s^k = f\left(\frac{s^k(t)}{s_e^k}\right), k = 1, \dots, n.$$

Then,

$$\begin{aligned} \frac{dV_s^k}{dt} &= f'\left(\frac{s^k(t)}{s_e^k}\right) \frac{1}{s_e^k} \frac{ds^k(t)}{dt} \\ &= \left(1 - \frac{s_e^k}{s^k(t)}\right) \frac{1}{s_e^k} [\mu p^k - \mu s^k - \alpha_k \sum_{j=1}^m \int_0^\infty \beta_j(\tau) i_j(t, \tau) s^k(t) d\tau] \\ &= \left(1 - \frac{s_e^k}{s^k(t)}\right) \frac{1}{s_e^k} [\mu(s_e^k - s^k(t)) + \alpha_k \sum_{j=1}^m \int_0^\infty \beta_j(\tau) [i_e^j(\tau) s_e^k - i^j(t, \tau) s^k(t)] d\tau] \\ &= -\mu \frac{(s^k(t) - s_e^k)^2}{s^k(t) s_e^k} + \alpha_k \sum_{j=1}^m \int_0^\infty \beta_j(\tau) i_e^j(\tau) \left[1 - \frac{i^j(t, \tau) s^k(t)}{i_e^j(\tau) s_e^k} - \frac{s_e^k}{s^k(t)} + \frac{i^j(t, \tau)}{i_e^j(\tau)}\right] d\tau. \end{aligned} \tag{4.7}$$

Let

$$V_i^k(t) = \int_0^\infty \Theta^k(\tau) f\left(\frac{i^j(t, \tau)}{i_e^j(\tau)}\right) d\tau,$$

where

$$\Theta^k(a) = \alpha_k \sum_{j=1}^m \int_a^\infty \beta_j(\tau) i_e^j(\tau) d\tau.$$

Then,

$$\begin{aligned}
\frac{dV_i^k}{dt} &= \frac{d}{dt} \int_0^\infty \Theta^k(\tau) f\left(\frac{i^j(t, \tau)}{i_e^j(\tau)}\right) d\tau \\
&= \frac{d}{dt} \int_0^\infty \Theta^k(\tau) f\left(\frac{b^j(t - \tau)}{i_e^j(0)}\right) d\tau \\
&= \frac{d}{dt} \int_{-\infty}^t \Theta^k(t - s) f\left(\frac{b^j(s)}{i_e^j(0)}\right) ds \\
&= \Theta^k(0) f\left(\frac{b^j(t)}{i_e^j(0)}\right) + \int_{-\infty}^t \Theta'^k(t - s) f\left(\frac{b^j(s)}{i_e^j(0)}\right) ds,
\end{aligned} \tag{4.8}$$

and thus

$$\frac{dV_i^k}{dt} = \Theta^k(0) f\left(\frac{i^j(t, 0)}{i_e^j(0)}\right) + \int_0^\infty \Theta'^k(\tau) f\left(\frac{i^j(t, \tau)}{i_e^j(\tau)}\right) d\tau. \tag{4.9}$$

Moreover, by the definition of  $\Theta^k$  we have

$$\Theta^k(0) f\left(\frac{i^j(t, 0)}{i_e^j(0)}\right) = \alpha_k \sum_{j=1}^m \int_0^\infty \beta_j(\tau) i_e^j(\tau) f\left(\frac{i^j(t, 0)}{i_e^j(0)}\right) d\tau. \tag{4.10}$$

Noticing additionally, that  $(\Theta^k)'(a) = -\alpha_k \sum_{j=1}^m \beta_j(a) i_e^j(a)$ , we may combine equation (4.9) and (4.10) to get

$$\frac{dV_i^k}{dt} = \alpha_k \sum_{j=1}^m \int_0^\infty \beta_j(\tau) i_e^j(\tau) \left[ f\left(\frac{i^j(t, 0)}{i_e^j(0)}\right) - f\left(\frac{i^j(t, \tau)}{i_e^j(\tau)}\right) \right] d\tau. \tag{4.11}$$

Substituting the expression of the function  $f$ , we obtain

$$\frac{dV_i^k}{dt} = \alpha_k \sum_{j=1}^m \int_0^\infty \beta_j(\tau) i_e^j(\tau) \left[ \frac{i^j(t, 0)}{i_e^j(0)} - \frac{i^j(t, \tau)}{i_e^j(\tau)} - \ln \frac{i^j(t, 0)}{i_e^j(0)} + \ln \frac{i^j(t, \tau)}{i_e^j(\tau)} \right] d\tau. \tag{4.12}$$

Let

$$V(t) = \sum_{k=1}^n V_s^k + \sum_{k=1}^n V_i^k.$$

Then, by combining (4.7) and (4.12) we have

$$\begin{aligned}
\frac{dV}{dt} &= -\mu \sum_{k=1}^n \frac{(s^k(t) - s_e^k)^2}{s^k(t) s_e^k} + \sum_{k=1}^n \alpha_k \sum_{j=1}^m \int_0^\infty \beta_j(\tau) i_e^j(\tau) \\
&\quad \left[ 1 - \frac{i^j(t, \tau) s^k(t)}{i_e^j(\tau) s_e^k} - \frac{s_e^k}{s^k(t)} + \frac{i^j(t, 0)}{i_e^j(0)} - \ln \frac{i^j(t, 0)}{i_e^j(0)} + \ln \frac{i^j(t, \tau)}{i_e^j(\tau)} \right] d\tau.
\end{aligned} \tag{4.13}$$

The objective now is to show that  $\frac{dV}{dt}$  is non-positive. In order to obtain this, we demonstrate that two of the terms above cancel out

$$\begin{aligned} & \sum_{j=1}^m \int_0^\infty \beta_j(\tau) i_e^j(\tau) \left[ \frac{i^j(t, 0)}{i_e^j(0)} - \frac{i^j(t, \tau) s^k(t)}{i_e^j(\tau) s_e^k} \right] d\tau \\ &= \frac{1}{s_e^k} \left[ \sum_{j=1}^m \int_0^\infty \beta_j(\tau) i_e^j(\tau) s_e^k d\tau \frac{i^j(t, 0)}{i_e^j(0)} - \sum_{j=1}^m \int_0^\infty \beta_j(\tau) i^j(t, \tau) s^k(t) d\tau \right]. \end{aligned}$$

We may multiply both sides of  $\sum_{j=1}^m \int_0^\infty \beta_j(\tau) i_e^j(\tau) s_e^k d\tau \frac{i^j(t, 0)}{i_e^j(0)} - \sum_{j=1}^m \int_0^\infty \beta_j(\tau) i^j(t, \tau) s^k(t) d\tau$

by  $\sum_{k=1}^n q_{kj} \alpha_k$  to obtain

$$\sum_{k=1}^n q_{kj} \alpha_k \sum_{j=1}^m \int_0^\infty \beta_j(\tau) i_e^j(\tau) s_e^k d\tau \frac{i^j(t, 0)}{i_e^j(0)} - \sum_{k=1}^n q_{kj} \alpha_k \sum_{j=1}^m \int_0^\infty \beta_j(\tau) i^j(t, \tau) s^k(t) d\tau.$$

Let  $q_{kj} = q_k$ ,

$$\begin{aligned} & \sum_{k=1}^n q_k \alpha_k \sum_{j=1}^m \int_0^\infty \beta_j(\tau) i_e^j(\tau) s_e^k d\tau \frac{i^j(t, 0)}{i_e^j(0)} - \sum_{k=1}^n q_k \alpha_k \sum_{j=1}^m \int_0^\infty \beta_j(\tau) i^j(t, \tau) s^k(t) d\tau \\ &= i_e^j(0) \frac{i^j(t, 0)}{i_e^j(0)} - i^j(t, 0) \\ &= 0. \end{aligned} \tag{4.14}$$

Using this expression to simplify equation (4.13), one gets

$$\frac{dV}{dt} = -\mu \sum_{k=1}^n \frac{(s^k(t) - s_e^k)^2}{s^k(t) s_e^k} + \sum_{k=1}^n \alpha_k \sum_{j=1}^m \beta_j(\tau) i_e^j(\tau) \left[ 1 - \frac{s_e^k}{s^k(t)} - \ln \frac{i^j(t, 0)}{i_e^j(0)} + \ln \frac{i^j(t, \tau)}{i_e^j(\tau)} \right] d\tau. \tag{4.15}$$

Noticing that  $\frac{i_e^j(0)}{i^j(t, 0)}$  is independent of  $\tau$ , we may multiply both sides of (4.14) by this quantity to obtain

$$\sum_{j=1}^m \int_0^\infty \beta_j(\tau) i_e^j(\tau) \left[ 1 - \frac{i^j(t, \tau) s^k(t)}{i_e^j(\tau) s_e^k} \frac{i_e^j(0)}{i^j(t, 0)} \right] d\tau = 0. \tag{4.16}$$

In addition, multiplying both sides of (4.16) by  $\alpha_k$  and summing for  $k$  from 1 to  $n$ , we may obtain

$$\sum_{k=1}^n \alpha_k \sum_{j=1}^m \int_0^\infty \beta_j(\tau) i_e^j(\tau) \left[ 1 - \frac{i^j(t, \tau) s^k(t)}{i_e^j(\tau) s_e^k} \frac{i_e^j(0)}{i^j(t, 0)} \right] d\tau = 0. \tag{4.17}$$

We now add (4.17) to (4.15), and also add and subtract  $\ln(\frac{s^k(t)}{s_e^k})$  to get

$$\frac{dV}{dt} = -\mu \sum_{k=1}^n \frac{(s^k(t) - s_e^k)^2}{s^k(t) s_e^k} + \sum_{k=1}^n \alpha_k \sum_{j=1}^m \int_0^\infty \beta_j(\tau) i_e^j(\tau) C^{j,k}(\tau) d\tau,$$

where

$$\begin{aligned}
C^{j,k}(\tau) &= 2 - \frac{i^j(t, \tau)}{i_e^j(\tau)} \frac{s^k(t)}{s_e^k} \frac{i_e^j(0)}{i^j(t, 0)} - \frac{s_e^k}{s^k(t)} - \ln \frac{i^j(t, 0)}{i_e^j(0)} + \ln \frac{i^j(t, \tau)}{i^j(a)} + \ln \frac{s^k(t)}{s_e^k} - \ln \frac{s^k(t)}{s_e^k} \\
&= (1 - \frac{s_e^k}{s^k(t)} + \ln \frac{s_e^k}{s^k(t)}) + (1 - \frac{i^j(t, \tau)}{i_e^j(\tau)} \frac{s^k(t)}{s_e^k} \frac{i_e^j(0)}{i^j(t, 0)} + \ln \frac{i^j(t, \tau)}{i_e^j(\tau)} \frac{s^k(t)}{s_e^k} \frac{i_e^j(0)}{i^j(t, 0)}) \\
&= -[f(\frac{s_e^k}{s^k(t)}) + f(\frac{i^j(t, \tau)}{i_e^j(\tau)} \frac{s^k(t)}{s_e^k} \frac{i_e^j(0)}{i^j(t, 0)})] \\
&\leq 0.
\end{aligned}$$

Thus,  $\frac{dV}{dt} \leq 0$ . Equality holds, if and only if

$$\frac{s_e^k}{s^k(t)} = 1, \text{ and } \frac{i^j(t, \tau)}{i_e^j(\tau)} \frac{i_e^j(0)}{i^j(t, 0)} = 1, \quad (4.18)$$

for all  $\tau \geq 0$ .

To complete the proof, we look for the largest invariant set  $M$  for which (4.16) holds. In  $M$ , we must have  $s^k(t) = s_e^k$  for all  $t$  and so we also have  $\frac{ds^k}{dt} = 0$ . Combining this result with (4.18), we get

$$\begin{aligned}
0 &= \mu p^k - \mu s_e^k - \alpha_k \sum_{j=1}^m \int_0^\infty \beta_j(\tau) i^j(t, \tau) d\tau s_e^k \\
&= \mu p^k - \mu s_e^k - \alpha_k \sum_{j=1}^m \int_0^\infty \beta_j(\tau) i^j(t, 0) B_j(a) e^{-\mu a} d\tau s_e^k \\
&= \mu p^k - \mu s_e^k - \frac{i^j(t, 0)}{i_e^j(0)} \alpha_k \sum_{j=1}^m \int_0^\infty \beta_j(\tau) i_e^j(0) B_j(a) e^{-\mu a} d\tau s_e^k \\
&= \mu p^k - \mu s_e^k - \frac{i^j(t, 0)}{i_e^j(0)} \alpha_k \sum_{j=1}^m \int_0^\infty \beta_j(\tau) i_e^j(\tau) d\tau s_e^k \\
&= \mu p^k - \mu s_e^k - \frac{i^j(t, 0)}{i_e^j(0)} (\mu p^k - \mu s_e^k) \\
&= (1 - \frac{i^j(t, 0)}{i_e^j(0)}) (\mu p^k - \mu s_e^k).
\end{aligned}$$

Since  $s_e^k$  is not equal to  $p^k$ , we must have  $i^j(t, 0) = i_e^j(0)$  and  $\frac{i^j(t, \tau)}{i_e^j(\tau)} = 1$  for all  $t$ .

**Theorem 4.6.** *Let Assumption 2.1 hold. Assume  $q_{kj} = q_k$ ,  $R_0 > 1$ . Then every solution that starts in  $A_0$  converges to the endemic equilibrium  $E_1^* = (s_e^k, i_e^j, r_e)$ ,  $k = 1, \dots, n$ ,  $j = 1, \dots, m$ .*

**Remark 4.7.** The endemic equilibrium is global asymptotically stable when  $m \leq n$  which is determined by the construction of Lyapunov function  $V$ . If  $m > n$ , the stability of the endemic equilibrium remains an open problem.

## 5 Discussion

In this article we have formulated compartmental differential susceptibility and differential infectivity models in various settings. The susceptibles and infectives are respectively divided into  $n$  and  $m$  subgroups based on their susceptibilities and infectivities. We consider the case where the total population size is constant. We study the transmission dynamics of the infection based on a variety of differential susceptibility and differential infectivity models. As typical in classical epidemiological models, there are two types of equilibria in our models. We derive an explicit formula for the reproductive number,  $R_0$ . For the model with standard incidence, we show that the infection-free equilibrium, whose component of infectives is zero, is globally asymptotically stable if  $R_0 < 1$ . If  $R_0 > 1$ , we further prove that there exists a unique endemic equilibrium with all components positive. Using a Lyapunov function on an invariant set, we further prove the global stability of the endemic equilibrium in the case  $m \leq n$ . The case  $m > n$  remains an open problem.

Recent mathematical epidemiology studies suggest that multiple endemic equilibria may exist and Hopf bifurcation may lead to periodicity for some endemic models [23]. Multi-strain transmission [24] may exhibit backward bifurcation and competitive exclusion. Our study, nevertheless, excludes such phenomena for the DS-DI epidemic models formulated in this paper. Our contribution here lies in extending the mathematical techniques for deriving global behavior of multigroup models structured by age-since-infection.

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