University of Florida

MAD4401 FINAL EXAM DECEMBER 15, 2003

Name:

ID #:

Instructor:

Directions: You have 2 hours to answer the following questions. You must show your work as neatly and clearly as possible and indicate the final answer clearly. You may NOT use a calculator.

Problem	Possible	Points
1	15	
2	20	
3	10	
4	15	
5	20	
6	20	
7	15	
8	15	
9	15	
10	20	
11	15	
12	20	
Total	200	

(1) (15 points) Use Hermite interpolation to find a polynomial of lowest degree satisfying

$$p(-1) = p'(-1) = 0, p(0) = 1, p(1) = p'(1) = 0.$$

Use the table below to compute the divided differences. Simplify your expression for $p(x)$ as much as possible.

x	f(x)	I DD	II DD	III DD	IV DD

(2) Consider the matrix

$$A = \begin{pmatrix} 2 & \alpha & -1 \\ \alpha & 2 & 1 \\ -1 & 1 & 4 \end{pmatrix}$$

(a) (5 points) Find all values of α so that the matrix A is strictly diagonally dominant.

(b) (7 points) Find all values of α so that the matrix A is singular.

(c) (8 points) Find all values of α so that the matrix A is positive definite.

(3) (a) (4 points) Find an interval $[a, b]$ that contains the positive root of the equation.

$$\cos x - 2xe^x = 0.$$

(b) (6 points) Find 3 different ways to rewrite the equation in (a) as a fixed point problem $x = g(x)$.

(4) (15 points) For the solution of the equation

$$x + \ln x = 0$$

which has a root $\alpha \approx 0.5$ the following fixed point iterations have been suggested.

(a) $p_{n+1} = -\ln p_n$

(b) $p_{n+1} = e^{-p_n}$

(c) $p_{n+1} = \dfrac{p_n + e^{-p_n}}{2}$

Without computing the sequences $p_0, p_1, \ldots$ decide which iterations will converge for appropriate p_0 and which will not converge. Justify your answer!

(5) Consider the linear system

$$A\mathbf{x} = \mathbf{b}$$

where

$$A = \begin{pmatrix} 2 & 3 & -1 \\ 4 & 4 & -1 \\ -2 & -3 & 4 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

(a) (10 points) Factor the matrix as $A = L \cdot U$ where L is a lower triangular matrix with ones along the diagonal and U is an upper triangular matrix.

(b) (10 points) If you were to solve the system $A\mathbf{x} = \mathbf{b}$ using the factorization $A = L \cdot U$ how many operations will you perform? Count separately additions/subtractions and multiplications/divisions.

(6) (20 points) Determine the cubic polynomial $p(x)$ in

$$s(x) = \begin{cases} p(x) & 0 \le x < 1 \\ (2-x)^3 & 1 \le x \le 2 \end{cases}$$

such that $s(0) = 0$ and $s(x)$ is a clamped cubic spline.

(7) Consider the integral

$$\int_0^{\pi} x^2 \cos x \, dx \tag{1}$$

(a) (10 points) Consider a quadrature rule of the form

$$\int_0^{\pi} x^2 f(x)\, dx \approx Af(0) + B\int_0^{\pi} f(x)\, dx.$$

Determine the constants A, B so that the quadrature formula has degree of exactness one.

(b) (5 points) Use the formula in part (a) to approximate the integral (1).

(8) You have been preforming Gaussian elimination on a matrix A and you have arrived at the following matrix:

$$\begin{pmatrix} 6 & 3 & 1 & 4 \\ 0 & 2 & -1 & 3 \\ 0 & 4 & 1 & 2 \\ 0 & 3 & 1 & -1 \end{pmatrix}$$

(a) (3 points) In the next step, if you are performing Gaussian elimination without pivoting what will the value of the pivot be?

(b) (5 points) In the next step, if you are performing Gaussian elimination with partial pivoting what will the value of the pivot be? Write down the matrix with the correct pivot in place.

(c) (7 points) In the next step, if you are performing Gaussian elimination with scaled partial pivoting what will the value of the pivot be? Write down the matrix with the correct pivot in place.

(9) Consider the integral

$$\int_1^2 x \ln x \, dx \tag{2}$$

(a) (10 points) Determine the number of intervals n and the step-size h necessary to approximate the integral (2) within 10^{-4} using Composite Simpson's rule.
Hint: Composite Simpson Rule is given by

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[f(a) + 4f(x_1) + 2f(x_2) + \cdots + 4f(x_{n-1}) + f(b)\right] - \frac{b-a}{180}h^4 f^{(4)}(\xi).$$

(b) (5 points) Set up Composite Simpson's rule to approximate integral (2) with $n = 4$ subdivisions.

(10) Consider the first order ordinary differential equation

$$y' = te^{3t} - 2y \qquad 0 \le t \le 1$$
$$y(0) = 0$$

whose actual solution (for reference) is $y = \frac{1}{5}te^{3t} - \frac{1}{25}e^{3t} + \frac{1}{25}e^{-2t}$.

(a) (5 points) Show that the function $f(t, y) = te^{3t} - 2y$ satisfies Lipscitz condition on the domain

$$\mathcal{D} = \{ (t, y) \mid 0 \le t \le 1, -\infty < y < \infty \}.$$

(b) (15 points) Compose Taylor's method of order two necessary for the solution of the above initial value problem with $h = 0.25$.
Hint: Taylor's method of order 2 is given by

$$w_0 = \alpha$$
$$w_{i+1} = w_i + hT^{(2)}(t_i, w_i) \qquad i = 0, 1, \ldots, N - 1$$

where $T^{(2)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2}f'(t_i, w_i)$.

(11) (15 points) Use least squares to fit a line

$$y = ax + b$$

through the points

x	0	1	2	3	4
Y	1	2	1	0	4

Hint: The normal equations are given by

$$Q_{11}a + Q_{12}b = R_1$$
$$Q_{12}a + nb = R_2$$

where $Q_{11} = \sum x_i^2$, $Q_{12} = \sum x_i$, $R_1 = \sum Y_i x_i$, $R_2 = \sum Y_i$. The solution of the system above is given by

$$a = \frac{nR_1 - Q_{12}R_2}{nQ_{11} - Q_{12}^2}$$
$$b = \frac{Q_{11}R_2 - R_1 Q_{12}}{nQ_{11} - Q_{12}^2}$$

(12) Consider a nonlinear equation which can be rewritten as a fixed point problem in the form $x = \pi + \frac{1}{2}\sin\frac{x}{2}$. This equation has a unique fixed point in the interval $[0, 2\pi]$.

(a) (10 points) Without computing the sequence $p_0, p_1 \ldots$ show that the iteration

$$p_{n+1} = \pi + \frac{1}{2}\sin\frac{p_n}{2}$$

converges for every p_0 in $[0, 2\pi]$.

(b) (4 points) What is the rate of convergence of the fixed point iteration in part (a)?

(c) (6 points) Estimate the number of iterations required to achieve accuracy 10^{-2}. Use the formula

$$|p_n - p| \leq k^n \max\{p_0 - a, b - p_0\}$$

and take $p_0 = \pi$.