3) Rate of convergence of sequences and functions

Ex. The sequence
\[ (\sin \left( \frac{1}{n^2} \right))^2 \]
converges to zero. Find the rate of convergence.

Solution: Since \( \sin x \leq x \)
\[ \sin \frac{1}{n^2} \leq \frac{1}{n^2} \]
\[ (\sin \frac{1}{n^2})^2 \leq \frac{1}{n^4} \]
⇒ The rate of convergence is \( O(\frac{1}{n^4}) \)

Ex. Find the rate of convergence of
\[ \lim_{h \to 0} \frac{e^h + e^{-h} - 2\cosh}{h} = 0 \]

Solution
\[ e^h = 1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \ldots \]
\[ e^{-h} = 1 - h + \frac{h^2}{2!} - \frac{h^3}{3!} + \ldots \]
\[ \cosh h = 1 - \frac{h^2}{2!} + \frac{h^4}{4!} + \ldots \]