

Imitation dynamics of vaccine decision-making behaviors based on the game theory*

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Abstract To investigate the imitation dynamics of vaccine uptake, age-structured model is introduced, based on a model proposed by Bauch. The model is derived under the assumption that the potential infection risk depends on the infection age. The existence and local stability of equilibria are analyzed. A Hopf bifurcation may occur from the endemic and vaccinator equilibrium. Our study shows that imitation behavior is one factor that destabilizes the system and brings about oscillations in the case when non-vaccinators imitate vaccinators more readily. Infection-age is another factor that produces limit cycles if the latency period is long enough. The results show how the prevalence of the infection changes with respect to the infection age.

Keywords: SIR model, game theory, decision-making, stability, Hopf bifurcation.

1 Introduction

Infectious diseases are one of the main enemies threatening human's health. Every year millions of people die as a result of various infectious diseases. Mathematical modeling plays an important role in studying and discovering the transmission mechanisms of important diseases. Ample theoretical work exists that investigates infectious diseases and obtains very valuable results for the control of epidemic spread. In some of the studies [1, 2, 3, 4, 5, 6] on childhood diseases, one eminent feature is that the transmission rates have salient differences since children have different immunities.

Vaccination has been widely considered as one of the effective methods to reducing the morbidity and mortality from infectious diseases. There are two key vaccination policies: voluntary vaccination and mandatory vaccination. There has been a vigorous debate about voluntary vaccination policy failing to protect population adequately. A rational vaccine decision-making is determined by various factors, such as perceived infection risk, potential side effects from vaccination, and vaccinating behaviors of other individuals. Because of the declining familiarity with the interplay between perceived infection risk and potential side effects from vaccination, parents have various reasons for avoiding the potential side effects for their children, instead relying on enough other children being vaccinated to

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provide herd immunity. Therefore, rational decision may lead to a reduced number of vaccine intakes. This is a free-riding dilemma between individuals and the public good. Game theory builds a bridge connecting the epidemic models with individual behaviors.

Vaccine decision-making based on game theory has been extensively investigated (see, for example, [7]-[23]). In these articles, it is usually assumed that individuals have full information for perceived infection risk and potential vaccine health risk. Under this assumption, rational vaccine decision-making will get the highest personal utilities, i.e., there exists a Nash equilibrium where no individuals could be better off by unilaterally changing to a different strategy [8, 9]. In [10], Xia and Liu employed a computational approach to characterize the impact of social influence on individuals' vaccination decision-making while in [11] they also investigated the impact of the two factors, information of the disease prevalence and the perceived vaccination risk, and fading coefficient of awareness spread. Recently, Xu and Cresmman [12] built a nonlinear epidemic model with the smoothed best response by game-theory based decisions on vaccination. Their results showed that if there is a perceived cost for vaccination, the smoothed best response is very effective in controlling the disease spread, but if vaccination is free, the best response is a good control. Zhang et al. [13] investigated the "double-edged sword" effect on public health conditions for rational decision-making. Shim et al used game dynamic models to gain insight into the decision-making between vaccine skeptics and vaccine believers [36] and the decision making with regard to antiviral intervention during an influenza pandemic [37], respectively.

The above-mentioned references focus on vaccine uptake based on the perfect information the parents mastered on perceived infection risk and potential vaccine side effects. In fact, the vaccine uptake behaviors evolve with respect to time. Individuals' decision on vaccination or non-vaccination is conducted by imitating others who appear to have adopted more successful strategies. The process is called "imitation dynamics" and it has been studied by some researchers. Bauch [14] investigated parents' vaccinating decisions for their children with the assumption that the susceptibles behave strategically in accordance with imitation dynamics and studied the dependence of epidemic prevalence and coverage of vaccination on these strategic decisions. Fu et al. [15] proposed an agent based model on social network with game theory to shed light on how imitation of peers shapes individual vaccination choices. d'Onofrio et al. [16] assumed that the perceived risk of vaccination is a function of the incidence and studied an SIR transmission model with dynamic vaccine demand based on an imitation mechanism. A common assumption in these existing vaccination models on imitation dynamics is that individuals are homogeneously mixed or heterogeneously mixed on social networks [17, 18, 38]. This can not capture the feature of different transmission rates for childhood diseases which we mentioned earlier. In order to address the imitation behaviors on vaccination decision-making, we develop an age-structured epidemic model based on the game theory. We assume that the transmission incidence rate varies according to the infection age and individuals act their behaviors with imperfect information about perceived infection risk and potential vaccine side effects. The payoff of the perceived infection risk is also dependent on the infection age. Parents imitate the vaccination decision-making of their neighbors and then adopt the vaccine policy.

The rest of the paper is organized as follows. Based on the game theory, an SIR epidemic model

with infection age is introduced in Section 2. We study the existence of equilibria and their local stability in Section 3. A local Hopf bifurcation may occur. Section 4 is devoted to the global stability of boundary equilibria based on the fluctuation lemma and the results for the case without vaccination. We also consider the persistence of the disease in Section 4. In Section 5, numerical simulations are provided to demonstrate the theoretical results. The paper concludes with discussion.

2 The model formulation

Chris T Bauch in [14] proposed an epidemic model to predict vaccinating behavior based on game theory. He divided the total population into three classes: the Susceptible $S(t)$, the Recovered $R(t)$, and the Infected $I(t)$ at time t . He assumed that all newborns are susceptible; all the groups decrease due to natural death; susceptible individuals are infected by infectious individuals and enter the infected group; infectious individuals exit the group due to natural death and recovery; recovered group increases due to recovered infectious individuals and the vaccinated newborns and decreases due to natural death; recovered individuals acquire permanent immunity from the disease or vaccination and never leave this group. The incidence rate used is in bilinear form. In order to understand a strategic interaction between individuals when they are deciding whether or not to vaccinate, he introduced a replicator equation for an evolution population game, and the payoff function varied according to time. In the replicator equation a proportion x of the children are vaccinated at birth. The equation formally reads as

$$\frac{dx}{dt} = kx(1-x)[-r_v + r_i m I]$$

where the vaccination cost is $-r_v$, non-vaccination cost is $r_i m I$. r_i denotes the perceived probability of suffering significant morbidity upon infection, m qualifies the sensitivity of vaccinating behavior to changes in prevalence, k denote the imitation rate. He proposed an epidemic model based on game theory given by the following equations.

$$\begin{cases} \frac{dS(t)}{dt} = \Lambda(1-x) - \beta S(t)I(t) - \mu S(t), \\ \frac{dI(t)}{dt} = \beta S(t)I(t) - (\mu + \gamma)I(t), \\ \frac{dR(t)}{dt} = \Lambda x + \gamma I(t) - \mu R(t), \\ \frac{dx(t)}{dt} = kx(1-x)[-r_v + r_i m I], \end{cases} \quad (2.1)$$

where Λ is the input rate, β is the transmission rate, μ is the natural death rate, γ is recovered rate. The incubation period of some childhood disease varies, for measles ranging from 7 to 21 days[33], for mumps ranging from 16 to 25 days[34], for pertussis ranging from 4 to 21 days [35]. It is easy to note that different diseases have different latency period. We introduce the infection age to describe this feature as well as the variability of infectivity shown by infectious individuals in many diseases. Therefore, we assume that the incidence rate depends on the infectious age a in the following way,

$$\lambda(t) = S(t) \int_0^\infty \beta(a) i(t, a) da.$$

where $\beta(a)$ is the transmission rate with infection age a .

For convenience, the perceived vaccination cost is $-r$; while the potential risk cost for non-vaccinator mainly depends on the epidemic transmission, so its cost is $-\int_0^\infty \beta(a)i(t, a)da$, or the payoff benefit is $\int_0^\infty \beta(a)i(t, a)da$. The payoff gain for vaccinator compared non-vaccinator is $\int_0^\infty \beta(a)i(t, a)da - r$. This is formally familiar with [12]. Then the derivation of x is described as a replicator equation

$$\frac{dx}{dt} = \delta x(1 - x) \left(\int_0^\infty \beta(a)i(t, a)da - r \right).$$

Based on the above assumptions, the model with game theory is given by the following system of ordinary and partial differential equations,

$$\begin{cases} \frac{dS(t)}{dt} = \Lambda(1 - x) - S(t) \int_0^\infty \beta(a)i(t, a)da - \mu S(t), \\ \frac{\partial i(t, a)}{\partial t} + \frac{\partial i(t, a)}{\partial a} = -(\mu + \gamma(a))i(t, a), \\ \frac{dR(t)}{dt} = \Lambda x + \int_0^\infty \gamma(a)i(t, a)da - \mu R(t), \\ \frac{dx(t)}{dt} = \delta x(1 - x) \left(\int_0^\infty \beta(a)i(t, a)da - r \right), \end{cases} \quad (2.2)$$

with the boundary condition

$$i(t, 0) = S(t) \int_0^\infty \beta(a)i(t, a)da, \quad t > 0$$

and initial conditions

$$S(0) = S_0 \geq 0, \quad i(0, a) = i_0(a) \in L_+^1, \quad R(0) = R_0 \geq 0, \quad x(0) = x_0 \in [0, 1],$$

where L_+^1 is the set of all integrable functions from $\mathbb{R}_+ = [0, \infty)$ into \mathbb{R}_+ . $\gamma(a)$ is the age-dependent recovery rate with infection age a , δ is the imitation rate which has the familiar biological meaning with k in (2.1).

Due to the biological meaning, Λ , μ , δ and r are all positive, and $\beta \neq 0$ and k belong to $C_{BU}(\mathbb{R}_+, \mathbb{R}_+)$ with β being not identically zero, where $C_{BU}(\mathbb{R}_+, \mathbb{R}_+)$ denotes the set of all bounded and uniformly continuous functions from \mathbb{R}_+ into \mathbb{R}_+ .

Note that the third equation of (2.2) is decoupled from the others. Then we can ignore it and only consider the following system,

$$\begin{cases} \frac{dS(t)}{dt} = \Lambda(1 - x) - S(t) \int_0^\infty \beta(a)i(t, a)da - \mu S(t), \\ \frac{\partial i(t, a)}{\partial t} + \frac{\partial i(t, a)}{\partial a} = -(\mu + \gamma(a))i(t, a), \\ \frac{dx(t)}{dt} = \delta x(1 - x) \left(\int_0^\infty \beta(a)i(t, a)da - r \right) \end{cases} \quad (2.3)$$

with the boundary condition

$$i(t, 0) = S(t) \int_0^\infty \beta(a)i(t, a)da, \quad t > 0$$

and initial conditions

$$S(0) = S_0 \geq 0, \quad i(0, a) = i_0(a) \in L_+^1, \quad x(0) = x_0 \in [0, 1].$$

Due to [24, 25], if i_0 satisfies the coupling equation

$$i(0, 0) = S_0 \int_0^\infty \beta(a) i_0(a) da$$

then (2.3) is well-posed. In what follows, whenever we refer to the solution of (2.3), the above assumptions on the boundary and initial conditions are satisfied.

Proposition 2.1. *Let $(S(t), i(t, a), x(t))$ be a solution of (2.3) with the maximal interval of existence $[0, \rho)$ (ρ is allowed to be ∞). Then $S(t) \geq 0$, $i(t, a) \geq 0$, and $x(t) \in [0, 1]$ for $t \in [0, \rho)$ and $a \in \mathbb{R}_+$.*

Proof. Firstly, from the third equation of (2.3), we have

$$x(t) = \frac{x_0 e^{\delta \int_0^t (\int_0^\infty \beta(a) i(s, a) da - r) ds}}{1 - x_0 + x_0 e^{\delta \int_0^t (\int_0^\infty \beta(a) i(s, a) da - r) ds}} \quad \text{for } t \in [0, \rho). \quad (2.4)$$

Since $x_0 \in [0, 1]$, it follows that $x(t) \in [0, 1]$ for $t \in [0, \rho)$.

Secondly, by the first equation of (2.3), we see that

$$S(t) = S_0 e^{-\mu t - \int_0^t \int_0^\infty \beta(a) i(s, a) da ds} + \Lambda \int_0^t (1 - x(s)) e^{-\mu(t-u) - \int_u^t \int_0^\infty \beta(a) i(s, a) da ds} du$$

for $t \in [0, \rho)$. Clearly, $S(t) \geq 0$ for $t \in [0, \rho)$ as $S_0 \geq 0$ and $x(t) \in [0, 1]$ for $t \in [0, \rho)$.

Now, for any $\xi \in [0, \rho)$, we claim that $i(t, a) \geq 0$ for $t \in [0, \xi]$ and $a \in \mathbb{R}_+$ (the idea of the proof is borrowed from Browne and Pilyugin [26]). By way of contradiction, assume that there exists $t_0 \in [0, \xi]$ and $a_0 \in \mathbb{R}_+$ such that $i(t_0, a_0) < 0$. Integrating the second equation of (2.3) with the boundary condition yields that, for $t \in [0, \xi]$ and $a \in \mathbb{R}_+$, we have

$$i(t, a) = \begin{cases} B(t - a)\pi(a), & t \geq a, \\ i_0(a - t) \frac{\pi(a)}{\pi(a - t)}, & t < a, \end{cases} \quad (2.5)$$

where $B(t) = i(t, 0) = S(t) \int_0^\infty \beta(a) i(t, a) da$ for $t \in [0, \xi]$ and $\pi(a) = e^{-\int_0^a k(s) ds}$ for $a \in \mathbb{R}_+$. Since $i_0 \in L_+^1$, we have $a_0 < t_0$ and $B(t_0 - a_0) < 0$. Let $t_* = \inf\{t \in [0, \xi] : B(t) < 0\}$. Then $t_* \in [0, \xi)$. By the continuity of $B(t)$ and the definition of t_* , we know that $B(t) \geq 0$ for $t \in [0, t_*)$ and $B(t_*) = 0$. Note that, with the help of (2.5),

$$\begin{aligned} B(t) &= S(t) \int_0^t \beta(a) B(t - a) \pi(a) da + S(t) \int_t^\infty \beta(a) i_0(a - t) \frac{\pi(a)}{\pi(a - t)} da \\ &= S(t) \int_0^t \beta(t - a) B(a) \pi(t - a) da + S(t) \int_t^\infty \beta(a) i_0(a - t) \frac{\pi(a)}{\pi(a - t)} da \end{aligned}$$

for $t \in [0, \xi]$. Since B is continuous on $[0, \xi]$ and $B(t_*) = 0$, there exists an $\varepsilon_1 \in (0, \xi - t_*)$ such that $-\frac{1}{2} \leq B(t) \leq \frac{1}{2}$ for $t \in [t_*, t_* + \varepsilon_1]$. Let $\hat{S} = \max\{S(t) : t \in [0, \xi]\}$ and $\bar{\beta} = \sup\{\beta(a) : a \in \mathbb{R}_+\}(> 0)$. Denote $\varepsilon = \min\{\varepsilon_1, \frac{1}{3\bar{\beta}(\hat{S}+1)}\}$. Let $Y = C([0, t_* + \varepsilon], \mathbb{R}_+)$ and $S = \{\varphi \in Y :$

$\varphi(t) = B(t)$ for $t \in [0, t_*]$ and $\varphi(t) \in [0, 1]$ for $t \in [t_*, t_* + \varepsilon]$. Then Y is a complete distance space with the supremum distance and S is a closed subset of Y . Define L on S by

$$(L\varphi)(t) = S(t) \int_0^t \beta(t-a)\varphi(a)\pi(a)da + S(t) \int_t^\infty \beta(a)i_0(a-t) \frac{\pi(a)}{\pi(a-t)} da.$$

Obviously, $L(S) \subset Y$. Note that $L\varphi|_{[0, t_*]} = B|_{[0, t_*]}$ for $\varphi \in S$ and $LB = B$. Then, for $\varphi \in S$ and $t \in [t_*, t_* + \varepsilon]$, we have

$$\begin{aligned} 0 &\leq (L\varphi)(t) = [(L\varphi)(t) - (LB)(t)] + (LB)(t) \\ &= [(L\varphi)(t) - (LB)(t)] + B(t) \\ &= S(t) \int_{t_*}^t \beta(t-a)(\varphi(a) - B(a))\pi(t-a)da + B(t) \\ &\leq \frac{3}{2}\hat{S}\bar{\beta}\varepsilon + \frac{1}{2} \\ &\leq 1 \end{aligned}$$

and this shows that L is a mapping from S into itself. Moreover, for $\varphi_1, \varphi \in S$, we have

$$\begin{aligned} \|L\varphi_1 - L\varphi_2\| &= \max\{|(L\varphi_1)(t) - (L\varphi_2)(t)| : t \in [t_*, t_* + \varepsilon]\} \\ &= \max\{|S(t) \int_{t_*}^t \beta(t-a)(\varphi_1(a) - \varphi_2(a))\pi(a)da| : t \in [t_*, t_* + \varepsilon]\} \\ &\leq \hat{S}\bar{\beta}\varepsilon\|\varphi_1 - \varphi_2\| \\ &< \frac{1}{3}\|\varphi_1 - \varphi_2\|, \end{aligned}$$

that is, L is a contraction mapping on S . By the Banach Fixed Point Theorem, there exists a unique $\hat{\varphi} \in S$ such that

$$\hat{\varphi}(t) = S(t) \int_0^t \beta(t-a)\hat{\varphi}(a)\pi(t-a)da + S(t) \int_t^\infty \beta(a)i_0(a-t) \frac{\pi(a)}{\pi(a-t)} da$$

for $t \in [0, t_* + \varepsilon]$. By the uniqueness of solutions, we have $B(t) = \hat{\varphi}(t)$ for $t \in [0, t_* + \varepsilon]$. This contradicts with the definition of t_* and hence we have proved the claim. Since ξ is arbitrary, we see that $i(t, a) \geq 0$ for $t \in [0, \rho)$ and $a \in \mathbb{R}_+$. This completes the proof. \square

Proposition 2.1 implies that $i(t, \cdot) \in L_+^1$ for $t \in [0, \rho)$. Setting

$$N(t) = S(t) + \int_0^\infty i(t, a)da$$

for $t \in [0, \rho)$. With the help of Proposition 2.1, we can obtain

$$\frac{dN(t)}{dt} \leq \Lambda - \mu N(t) \quad (2.6)$$

for $t \in [0, \rho)$, which implies that $\limsup_{t \rightarrow \rho^-} N(t) < \infty$. Therefore, the maximal interval of existence for every solution of (2.3) is \mathbb{R}_+ . Let $\Phi : \mathbb{R}_+ \times (\mathbb{R}_+ \times L_+^1 \times [0, 1]) \rightarrow \mathbb{R}_+ \times L_+^1 \times [0, 1]$ be the solution semiflow associated with (2.3), that is,

$$\Phi(t, (S_0, i_0, x_0)) = (S(t), i(t, \cdot), x(t)) \quad \text{for } (S_0, i_0, x_0) \in \mathbb{R}_+ \times L_+^1 \times [0, 1].$$

Note that, by (2.6), $\limsup_{t \rightarrow \infty} N(t) \leq \frac{\Lambda}{\mu}$. Moreover, if $N(t_0) \leq \frac{\Lambda}{\mu}$ for some $t_0 \in \mathbb{R}_+$ then $N(t) \leq \frac{\Lambda}{\mu}$ for all $t \geq t_0$. Define

$$\Gamma = \{(S, i, x) \in \mathbb{R}_+ \times L_+^1 \times [0, 1] : S + \int_0^\infty i(a) da \leq \frac{\Lambda}{\mu}\}.$$

Then we have proved the following result.

Proposition 2.2. Γ is an attractive and positively invariant set for (2.3).

To end this section, we mention that, by (2.4), both sets $\{(S, i, x) \in \Gamma : x = 0\}$ and $\{(S, i, x) \in \Gamma : x = 1\}$ are also positively invariant subsets of (2.3). It is clear that for solutions in $\{(S, i, x) \in \Gamma : x = 1\}$ we have $\lim_{t \rightarrow \infty} (S(t), i(t, a), x(t)) = (0, 0, 1)$ for $a \in \mathbb{R}_+$. On the other invariant set $\{(S, i, x) \in \Gamma : x = 0\}$, (2.3) reduces to the case without vaccination, that is,

$$\begin{cases} \frac{dS(t)}{dt} = \Lambda - S(t) \int_0^\infty \beta(a) i(t, a) da - \mu S(t), \\ \frac{\partial i(t, a)}{\partial t} + \frac{\partial i(t, a)}{\partial a} = -(mu + \gamma(a)) i(t, a). \end{cases} \quad (2.7)$$

This model has been studied by Magal *et al.* [27] and their main results are as follows.

Theorem 2.3 ([27, Theorem 1.2 and Theorem 1.3]). (i) If $\frac{\Lambda \int_0^\infty \beta(a) \pi(a) da}{\mu} \leq 1$ then the disease free equilibrium $(\frac{\Lambda}{\mu}, 0)$ is globally asymptotically stable for the semiflow generated by (2.7).

(ii) Let

$$\bar{a} = \sup\{a \geq 0 : \beta(a) > 0\}$$

and

$$M_0 = \mathbb{R}_+ \times \{i \in L_+^1 : \int_0^{\bar{a}} i(a) da > 0\} \quad \text{and} \quad \partial M_0 = (\mathbb{R}_+ \times L_+^1) \setminus M_0.$$

Assume $\frac{\Lambda \int_0^\infty \beta(a) \pi(a) da}{\mu} \leq 1$. Then every solution of (2.7) with the initial value in ∂M_0 (respectively in M_0) stays in ∂M_0 (respectively in M_0). Moreover, each solution with initial value in ∂M_0 converges to $(\frac{\Lambda}{\mu}, 0)$. Furthermore, every solution with an initial value in M_0 converges to the endemic equilibrium $(\frac{1}{\int_0^\infty \beta(a) \pi(a) da}, (\Lambda - \frac{\mu}{\int_0^\infty \beta(a) \pi(a) da}) \pi(a))$, which is locally asymptotically stable.

The attractivity of the endemic equilibrium in Theorem 2.3(ii) is established by the approach of Lyapunov functionals. We also mention that Theorem 2.3 will play an important role in dealing with the attractivity of the boundary equilibria of (2.3).

3 Existence of equilibria and their local stability

Recall that $\pi(a) = e^{-\int_0^a k(s) ds}$ for $a \in \mathbb{R}_+$. Define

$$K = \int_0^\infty \beta(a) \pi(a) da$$

and

$$\mathcal{R}_0 = \frac{\Lambda K}{\mu}.$$

In fact, \mathcal{R}_0 is called the basic reproduction number and it denotes the average number of cases that one typical infected individual can produce during the infectious period. As we will see soon, the structure of equilibria of (2.3) depends on the values of \mathcal{R}_0 .

$(S^*, i^*, x^*) \in \mathbb{R}_+ \times L_+^1 \times [0, 1]$ is an equilibrium of (2.3) if it satisfies

$$\begin{cases} 0 = \Lambda(1 - x^*) - S^* \int_0^\infty \beta(a) i^*(a) da - \mu S^*, \\ \frac{di^*(a)}{da} = -(mu + \gamma(a)) i^*(a), \\ i^*(0) = S^* \int_0^\infty \beta(a) i^*(a) da, \\ 0 = \delta x^* (1 - x^*) (\int_0^\infty \beta(a) i^*(a) da - r). \end{cases} \quad (3.1)$$

By direct computation, we easily see that (3.1) has four equilibria, $E_1 = (0, 0, 1)$, $E_2 = (\frac{\Lambda}{\mu}, 0, 0)$, $E_3 = (S^*, i_1^*(0)\pi(a), 0)$, $E_4 = (S^*, i^*(0)\pi(a), x^*)$, where

$$S^* = \frac{1}{K}, \quad i_1^*(0) = \frac{\mu}{K}(\mathcal{R}_0 - 1), \quad i^*(0) = \frac{r}{K}, \quad x^* = \frac{\mu(\mathcal{R}_0 - 1) - r}{\Lambda K}.$$

The existence of equilibria of (2.3) is summarized below.

Theorem 3.1. (i) If $\mathcal{R}_0 \leq 1$ then (2.3) has two equilibria E_1 and E_2 .

(ii) If $1 < \mathcal{R}_0 \leq 1 + \frac{r}{\mu}$ then (2.3) has three equilibria E_1 , E_2 , and E_3 .

(iii) If $\mathcal{R}_0 > 1 + \frac{r}{\mu}$ then (2.3) has four equilibria E_1 , E_2 , E_3 , and E_4 .

From the biological perspective, E_1 is the disease-free and pure vaccinator strategy equilibrium, E_2 is the disease-free and non-vaccinator strategy equilibrium, E_3 is the endemic and non-vaccinator strategy equilibrium, and E_4 is the endemic and vaccinator strategy equilibrium.

Let $\bar{E}^* = (\bar{S}^*, \bar{i}^*, \bar{x}^*)$ be an equilibrium of (2.3). Then the linearized system at \bar{E}^* is

$$\begin{cases} \frac{dS(t)}{dt} = -\Lambda x - \bar{S}^* \int_0^\infty \beta(a) i(t, a) da - S \int_0^\infty \beta(a) \bar{i}^*(a) da - \mu S, \\ \frac{\partial i(t, a)}{\partial t} + \frac{\partial i(t, a)}{\partial a} = -(mu + \gamma(a)) i(t, a), \\ i(t, 0) = S(t) \int_0^\infty \beta(a) \bar{i}^*(a) da + \bar{S}^* \int_0^\infty \beta(a) i(t, a) da, \\ \frac{dx(t)}{dt} = \delta(1 - 2\bar{x}^*)x(\int_0^\infty \beta(a) \bar{i}^*(a) da - r) + \delta\bar{x}^*(1 - \bar{x}^*) \int_0^\infty \beta(a) i(t, a) da. \end{cases}$$

Letting $S(t) = s_0 e^{\lambda t}$, $i(t, a) = y(a) e^{\lambda t}$, and $x(t) = x_0 e^{\lambda t}$ leads to the characteristic equation at \bar{E}^* , which is

$$0 = \begin{vmatrix} \lambda + \mu + \int_0^\infty \beta(a) \bar{i}^*(a) da & \bar{S}^* \hat{K}(\lambda) & \Lambda \\ \int_0^\infty \beta(a) \bar{i}^*(a) da & \bar{S}^* \hat{K}(\lambda) - 1 & 0 \\ 0 & -\delta\bar{x}^*(1 - \bar{x}^*) \hat{K}(\lambda) & \lambda - \delta(1 - 2\bar{x}^*) (\int_0^\infty \beta(a) \bar{i}^*(a) da - r) \end{vmatrix},$$

where $\hat{K}(\lambda) = \int_0^\infty \beta(a) \pi(a) e^{-\lambda a} da$. Then we can get the local stability of E_1 , E_2 , and E_3 as follows.

Theorem 3.2. (i) The disease free and pure vaccinator equilibrium E_1 is always unstable.

(ii) The disease free and non-vaccinator equilibrium E_2 is locally asymptotically stable if $\mathcal{R}_0 < 1$ and it is unstable if $\mathcal{R}_0 > 1$.

- (iii) The endemic and non-vaccinator equilibrium E_3 is locally asymptotically stable if $1 < \mathcal{R}_0 < 1 + \frac{r}{\mu}$ and it is unstable if $\mathcal{R}_0 > 1 + \frac{r}{\mu}$.

Proof. (i) The characteristic equation for E_1 is

$$\begin{vmatrix} \lambda + \mu & 0 & \Lambda \\ 0 & -1 & 0 \\ 0 & 0 & \lambda - \delta r \end{vmatrix} = (\lambda + \mu)(\lambda - \delta r) = 0,$$

which has a positive root δr . This implies that E_1 is unstable.

(ii) The characteristic equation at E_2 is

$$\begin{vmatrix} \lambda + \mu & \frac{\Lambda}{\mu} \hat{K}(\lambda) & \Lambda \\ 0 & \frac{\Lambda}{\mu} \hat{K}(\lambda) - 1 & 0 \\ 0 & 0 & \lambda + \delta r \end{vmatrix} = (\lambda + \mu)(\lambda + \delta r) \left(\frac{\Lambda}{\mu} \hat{K}(\lambda) - 1 \right) = 0.$$

Besides the two negative roots $-\mu$ and $-\delta r$, the other roots are given by the equation

$$\frac{\Lambda}{\mu} \hat{K}(\lambda) = 1. \quad (3.2)$$

Since $\frac{\Lambda}{\mu} \hat{K}(0) = \mathcal{R}_0$ and $\lim_{\lambda \rightarrow \infty} \hat{K}(\lambda) = 0$, it follows that (3.2) has a positive root if $\mathcal{R}_0 > 1$. If $\mathcal{R}_0 < 1$, we claim that all roots of (3.2) have negative real parts. Otherwise, if λ_0 is a root of (3.2) with nonnegative real part, then

$$1 = \left| \frac{\Lambda}{\mu} \hat{K}(\lambda) \right| = \left| \frac{\Lambda}{\mu} \int_0^\infty \beta(a) \pi(a) e^{-\lambda_0 a} da \right| \leq \frac{\Lambda}{\mu} \int_0^\infty \beta(a) \pi(a) da = \frac{\Lambda}{\mu} K = \mathcal{R}_0 < 1,$$

a contradiction. This proves the claim. Therefore, E_2 is stable if $\mathcal{R}_0 < 1$ and it is unstable if $\mathcal{R}_0 > 1$.

(iii) The characteristic equation at E_3 is

$$\begin{aligned} 0 &= \begin{vmatrix} \lambda + \mu + i_1^*(0)K & S^* \hat{K}(\lambda) & \Lambda \\ i_1^*(0)K & S^* \hat{K}(\lambda) - 1 & 0 \\ 0 & 0 & \lambda - \delta i_1^*(0)K + \delta r \end{vmatrix} \\ &= (\lambda - \delta i_1^*(0)K + \delta r)[(\lambda + \mu)(S^* \hat{K}(\lambda) - 1) - i_1^*(0)K]. \end{aligned}$$

First, the root of $\lambda - \delta i_1^*(0)K + \delta r = 0$ is $\delta i_1^*(0)K - \delta r = \delta \mu(\mathcal{R}_0 - 1 - \frac{r}{\mu})$, which is positive (respectively, negative) if $\mathcal{R}_0 > 1 + \frac{r}{\mu}$ (respectively, $\mathcal{R}_0 < 1 + \frac{r}{\mu}$). Second, we claim that $0 = (\lambda + \mu)(S^* \hat{K}(\lambda) - 1) - i_1^*(0)K$ has no root with nonnegative real part. By contradiction, suppose it has a root λ_1 with nonnegative real part. Then

$$0 = (\lambda_1 + \mu)(S^* \hat{K}(\lambda_1) - 1) - i_1^*(0)K = \lambda_1 + \mu \mathcal{R}_0 - \frac{\lambda_1 + \mu}{K} \hat{K}(\lambda_1).$$

It follows that

$$|\lambda_1 + \mu \mathcal{R}_0| = \left| \frac{\lambda_1 + \mu}{K} \hat{K}(\lambda_1) \right| \leq |\lambda_1 + \mu|,$$

which is a contradiction since

$$|\lambda + \mu \mathcal{R}_0|^2 - |\lambda_1 + \mu|^2 = 2\operatorname{Re}(\lambda_1)\mu(\mathcal{R}_0 - 1) + \mu^2(\mathcal{R}_0^2 - 1)$$

and $\operatorname{Re}(\lambda_1) \geq 0$ and $\mathcal{R}_0 > 1$. This proves the claim. To summarize, we have proved (iii). \square

Theorem 3.2 tells us that the disease free and pure vaccinator equilibrium E_1 is always unstable. This means that if the level of the vaccinated children is very high then the unvaccinated children have no incentives to vaccinate since the herd immunity can protect them and they do not care about the potential risk from vaccination.

The analysis for the stability of E_4 is not so easy. In fact, the characteristic equation at E_4 is

$$\begin{aligned}
0 &= \begin{vmatrix} \lambda + \mu + i^*(0)K & S^* \hat{K}(\lambda) & \Lambda \\ i^*(0)K & S^* \hat{K}(\lambda) - 1 & 0 \\ 0 & -\delta x^*(1 - x^*) \hat{K}(\lambda) & \lambda - \delta(1 - 2x^*)(i^*(0)K - r) \end{vmatrix} \\
&= \begin{vmatrix} \lambda + \mu & 1 & \Lambda \\ r & \frac{\hat{K}(\lambda)}{K} - 1 & 0 \\ 0 & -\delta x^*(1 - x^*) \hat{K}(\lambda) & \lambda \end{vmatrix} \\
&= \lambda \left[(\lambda + \mu) \left(\frac{\hat{K}(\lambda)}{K} - 1 \right) - r \right] - C \hat{K}(\lambda),
\end{aligned} \tag{3.3}$$

where $C = \Lambda r \delta x^*(1 - x^*)$. Obviously, 0 is not a root to (3.3) when $\mathcal{R}_0 > 1 + \frac{r}{\mu}$. Then (3.3) is equivalent to

$$\frac{\hat{K}(\lambda)}{K} = 1 + \frac{r}{\lambda + \mu} + \frac{C}{\lambda(\lambda + \mu)} \hat{K}(\lambda).$$

It follows easily that (3.3) has no nonnegative real roots. However, it is hard to see whether (3.3) has roots with nonnegative real part or not. Actually, it may have roots with nonnegative real parts as we will see soon.

In order to have a clear picture of the dynamics of (2.3), we make the following further assumption.

Assumption 3.1. Assume that the transmission rate and the transfer rate are

$$\beta(a) = \begin{cases} \beta & \text{if } a \geq \tau \\ 0 & \text{if } a < \tau \end{cases} \quad \text{and} \quad \gamma(a) = \begin{cases} \gamma & \text{if } a \geq \tau \\ 0 & \text{if } a < \tau. \end{cases}$$

Assumption 3.1 means that when the infection age is larger than τ the infected is also infectious and the infectivity is the same for all such infected individuals. We call τ the latent period. Assumption 3.1 also implies that the self-healing only occurs when the infected become infectious. If we add a constant recovery rate for infection age $a \in [0, \tau)$ then the analysis is reasonably simple and we can still gain further insight into the dynamics. However, if we assume that the recovery rates differ at an infection age other than the latent period τ then the analysis will be very complicated. In the following we analyze the role played by τ .

Under Assumption 3.1, $\hat{K}(\lambda) = \frac{\beta e^{-(\lambda + \mu)\tau}}{\lambda + \mu + \gamma}$. It follows that

$$\mathcal{R}_0 = \frac{\Lambda \beta e^{-\mu\tau}}{\mu(\mu + \gamma)}.$$

Let $\tau_{\min} = \frac{1}{\mu} \ln \frac{\Lambda \beta}{(\mu + r)(\mu + \gamma)}$ and $\tau_{\max} = \frac{1}{\mu} \ln \frac{\Lambda \beta}{\mu(\mu + \gamma)}$. Then $\mathcal{R}_0 > 1$ if and only if $\frac{\Lambda \beta}{\mu(\mu + \gamma)} > 1$ and $0 \leq \tau < \tau_{\max}$; $\mathcal{R}_0 > 1 + \frac{r}{\mu}$ if and only if $\frac{\Lambda \beta}{(\mu + r)(\mu + \gamma)} > 1$ and $0 \leq \tau < \tau_{\min}$. Moreover,

$$E_4 = \left(\frac{(\mu + \gamma)e^{\mu\tau}}{\beta}, \frac{r(\mu + \gamma)e^{\mu\tau}}{\beta} \pi(a), 1 - \frac{(r + \mu)(\mu + \gamma)e^{\mu\tau}}{\Lambda \beta} \right)$$

and (3.3) becomes

$$\lambda(\lambda + \mu + \gamma)(\lambda + \mu + r) = [(\mu + \gamma)\lambda(\lambda + \mu) - D(\tau)]e^{-\lambda\tau}, \quad (3.4)$$

where $D(\tau) = \beta C e^{-\mu\tau} = \frac{r\delta(r+\mu)(\mu+\gamma)(\Lambda\beta-(r+\mu)(\mu+\gamma)e^{\mu\tau})}{\Lambda\beta}$.

Assume $\tau = 0$. Then (3.4) reduces to

$$\lambda^3 + (\mu + r)\lambda^2 + r(\mu + \gamma)\lambda + D(0) = 0.$$

Note that $r(\mu + \gamma)(\mu + r) - D(0) = r(\mu + \gamma)(\mu + r)(1 - \frac{\delta(\Lambda\beta-(\mu+r)(\mu+\gamma))}{\Lambda\beta})$. Due to the Routh-Hurwitz Criterion, E_4 is locally asymptotically stable if $\frac{\delta(\Lambda\beta-(\mu+r)(\mu+\gamma))}{\Lambda\beta} < 1$ and unstable otherwise. By Theorem 3.2, we know that E_1 , E_2 , and E_3 all are unstable if $\mathcal{R}_0 > 1 + \frac{r}{\mu}$. Therefore, we make one more assumption.

Assumption 3.2. $\delta < \frac{\Lambda\beta}{\Lambda\beta-(\mu+\gamma)(\mu+r)}$, i.e., $\delta x^* < 1$. This last assumption can be interpreted as a condition that the total number of non-vaccinators imitating vaccinators should be small enough.

Next, we consider the possibility of the stability switching for system (2.3). Note that (3.4) is a special case of the transcendental equation considered by Beretta and Kuang [28], where they provided practical guidelines that combine graphical information with analytical work to effectively study local stability. The theory was also illustrated by them with first order and second order characteristic equations. Here is an application to a third order characteristic equation. Let

$$\begin{aligned} P(\lambda, \tau) &= \lambda(\lambda + \mu + \gamma)(\lambda + \mu + r), \\ Q(\lambda, \tau) &= D(\tau) - (\mu + \gamma)\lambda(\lambda + \mu). \end{aligned}$$

Then one can easily verify assumptions (i)–(iv) in Beretta and Kaung [28, p. 1145]. Under Assumptions 3.1 and 3.2, a stability change at E_4 can only happen when there are eigenvalues crossing the imaginary axis from left to right. Therefore, we seek a pair of purely imaginary eigenvalues $\lambda = \pm iw$ with $w > 0$ for some $0 < \tau < \tau_{\min}$ since 0 is not an eigenvalue. Substituting $\lambda = iw$ into (3.4) (we only need to consider one of the roots) and separating the real and imaginary parts yields

$$\begin{aligned} -(2\mu + r + \gamma)w^2 &= -((\mu + \gamma)w^2 + D(\tau)) \cos w\tau + \mu(\mu + \gamma)w \sin w\tau, \\ \omega[(\mu + \gamma)(\mu + r) - \omega^2] &= ((\mu + \gamma)w^2 + D(\tau)) \sin w\tau + \mu(\mu + \gamma)w \cos w\tau. \end{aligned} \quad (3.5)$$

Then

$$\begin{aligned} \sin w\tau &= -\frac{w[(\mu+\gamma)w^4+(D(\tau)+\mu^2(\mu+\gamma)+\mu(\mu+\gamma)r-(\mu+\gamma)^2r)w^2-D(\tau)(\mu+\gamma)(\mu+r)]}{[\mu(\mu+\gamma)w]^2+[(\mu+\gamma)w^2+D(\tau)]^2}, \\ \cos w\tau &= \frac{w^2[(\mu+\gamma)(\mu+\gamma+r)w^2+D(\tau)(2\mu+\gamma+r)+\mu(\mu+\gamma)^2(\mu+r)]}{[\mu(\mu+\gamma)w]^2+[(\mu+\gamma)w^2+D(\tau)]^2}. \end{aligned} \quad (3.6)$$

We square both sides of the equations in (3.5) and add them up to obtain

$$F(w, \tau) \triangleq w^6 + (\mu + r)^2 w^4 + (\mu + \gamma)[r(\mu + \gamma)(r + 2\mu) - 2D(\tau)]w^2 - D(\tau)^2 = 0,$$

or

$$z^3 + (\mu + r)^2 z^2 + (\mu + \gamma)[r(\mu + \gamma)(r + 2\mu) - 2D(\tau)]z - D(\tau)^2 = 0 \quad (3.7)$$

with $z = w^2$. According to Descartes' Rule of Signs, equation (3.7) has at most a positive root. On the other hand, it is easy to see that (3.7) has a positive root. Therefore, equation (3.7) has a unique positive root, denoted by $w_0^2(\tau)$ with $w_0(\tau) > 0$. As $w_0^2(\tau)$ is a simple root of (3.7), we have

$$F_w(w_0(\tau), \tau) > 0.$$

Let $\theta(\tau) \in [0, 2\pi]$ be the solution of

$$\begin{aligned}\sin \theta(\tau) &= -\frac{w_0(\tau)[(\mu+\gamma)w_0^4(\tau)+(D(\tau)+\mu^2(\mu+\gamma)+\mu(\mu+\gamma)r-(\mu+\gamma)^2r)w_0^2(\tau)-D(\tau)(\mu+\gamma)(\mu+r)]}{[\mu(\mu+\gamma)w_0(\tau)]^2+[(\mu+\gamma)w_0^2(\tau)+D(\tau)]^2}, \\ \cos \theta(\tau) &= \frac{w_0^2(\tau)[(\mu+\gamma)(\mu+\gamma+r)w_0^2(\tau)+D(\tau)(2\mu+\gamma+r)+\mu(\mu+\gamma)^2(\mu+r)]}{[\mu(\mu+\gamma)w_0(\tau)]^2+[(\mu+\gamma)w_0^2(\tau)+D(\tau)]^2}.\end{aligned}$$

(obtained by substituting $w_0(\tau)$ into the right hand sides of (3.6)). For any $n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$, define $S_n : [0, \tau_{\min}) \rightarrow \mathbb{R}$ by

$$S_n(\tau) = \tau - \frac{\theta(\tau) + 2n\pi}{w_0(\tau)} \quad \text{for } \tau \in [0, \tau_{\min}).$$

The following result is deduced from Theorem 2.2 of Beretta and Kuang [28].

Proposition 3.3. *Suppose that there exist $n_0 \in \mathbb{N}_0$ and $\tau^* \in (0, \tau_{\min})$ such that $S_{n_0}(\tau^*) = 0$. Then (3.4) has a pair of simple conjugate pure imaginary roots $\lambda_{\pm}(\tau^*) = \pm iw_0(\tau^*)$ at $\tau = \tau^*$. Moreover, the pair of simple conjugate imaginary roots crosses the imaginary axis from left to right if $s_{n_0}(\tau^*) > 0$ and crosses the imaginary axis from right to left if $s_{n_0}(\tau^*) < 0$, where*

$$s_{n_0}(\tau^*) = \text{sign} \left(\frac{d\text{Re}\lambda(\tau^*)}{d\tau} \right) = \text{sign} \left(\frac{dS_{n_0}(\tau^*)}{d\tau} \right).$$

Denote

$$\mathbb{S} = \{\tau \in (0, \tau_{\min}) : S_{n(\tau)}(\tau) = 0 \text{ for some } n(\tau) \in \mathbb{N}_0\}.$$

The set \mathbb{S} is finite though it is a little bit long and difficult to show here. If $\mathbb{S} \neq \emptyset$, let $\mathbb{S} = \{\tau_0, \tau_1, \dots, \tau_{n_0}\}$ for some $n_0 \in \mathbb{N}_0$ with $\tau_0 < \tau_1 < \dots < \tau_{n_0}$. It is easy to deduce $s_{n(\tau_0)}(\tau_0) \geq 0$ as (2.3) is stable when τ is small enough. The following result follows from Proposition 3.3 and the above discussion.

Theorem 3.4. *Suppose Assumptions 3.1 and 3.2 hold and $\tau < \tau_{\min}$. Then the following statements are true.*

- (i) *If $\mathbb{S} = \emptyset$ then the endemic and vaccinator equilibrium E_4 is locally asymptotically stable for all $\tau \in [0, \tau_{\min})$.*
- (ii) *If $\mathbb{S} = \{\tau_0, \tau_1, \dots, \tau_{n_0}\}$ with $\tau_0 < \tau_1 < \dots < \tau_{n_0}$ then the endemic and vaccinator equilibrium E_4 is locally asymptotically stable for all $\tau \in [0, \tau_0)$. Moreover, if $s_{n(\tau_0)}(\tau_0) > 0$ then a Hopf bifurcation occurs at $\tau = \tau_0$.*

Under the condition in Theorem 3.4, there may be a stability switch for E_4 . A stability switch may occur if there exists $m_0 \leq n_0$ such that $\sum_{n=0}^{m_0} s(\tau_n) \leq 0$.

4 The attractivity of boundary equilibria and disease persistence

In this section, we first study the attractivity of the boundary equilibria E_2 and E_3 by applying Theorem 2.3 and the comparison principle.

To establish the attractivity of E_2 , we need the fluctuation lemma. For a function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}$, we denote

$$\varphi_\infty = \liminf_{t \rightarrow \infty} \varphi(t) \quad \text{and} \quad \varphi^\infty = \limsup_{t \rightarrow \infty} \varphi(t).$$

Lemma 4.1 (Fluctuation Lemma [29]). *Let $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a bounded and continuously differentiable function. Then there exist sequences $\{s_n\}$ and $\{t_n\}$ such that $s_n \rightarrow \infty$, $t_n \rightarrow \infty$, $\varphi(s_n) \rightarrow \varphi_\infty$, $\varphi'(s_n) \rightarrow 0$, $\varphi(t_n) \rightarrow \varphi^\infty$, and $\varphi'(t_n) \rightarrow 0$ as $n \rightarrow \infty$.*

Theorem 4.2. *Suppose that $\mathcal{R}_0 \leq 1$. Then the equilibrium E_2 attracts all solutions of (2.3) with $(S_0, i_0, x_0) \in \{(S, i, x) \in \Gamma : x \in [0, 1]\}$.*

Proof. Let $(S(t), i(t, a), x(t))$ be any solution of (2.3) with initial condition and boundary conditions from $\{(S, i, x) \in \Gamma : x \in [0, 1]\}$. Then $x(t) \in [0, 1]$ for $t \in \mathbb{R}_+$ and hence $(S(t), i(t, x))$ satisfies

$$\begin{cases} \frac{dS(t)}{dt} \leq \Lambda - S(t) \int_0^\infty \beta(a) i(t, a) da - \mu S(t), \\ \frac{\partial i(t, a)}{\partial t} + \frac{\partial i(t, a)}{\partial a} = -(mu + \gamma(a)) i(t, a). \end{cases}$$

Let $(\bar{S}(t), \bar{i}(t, a))$ be the solution of the auxiliary system

$$\begin{cases} \frac{dS(t)}{dt} = \Lambda - S(t) \int_0^\infty \beta(a) \bar{i}(t, a) da - \mu S(t) \\ \frac{\partial \bar{i}(t, a)}{\partial t} + \frac{\partial \bar{i}(t, a)}{\partial a} = -(mu + \gamma(a)) \bar{i}(t, a) \end{cases} \quad (4.1)$$

with $(\bar{S}_0, \bar{i}_0) = (S_0, i_0)$ and $\bar{i}(t, 0) = \bar{S}(t) \int_0^\infty \beta(a) \bar{i}(t, a) da$. With the help of the comparison principle, it is easy to see that $0 \leq S(t) \leq \bar{S}(t)$ and $i(t, a) \leq \bar{i}(t, a)$ for $(t, a) \in \mathbb{R}_+^2$. Since $\mathcal{R}_0 \leq 1$, it follows from Theorem 2.3 that $(\bar{S}(t), \bar{i}(t, a))$ converges to $(\frac{\Lambda}{\mu}, 0)$. Then

$$0 \leq \limsup_{t \rightarrow \infty} \int_0^\infty i(t, a) da \leq \limsup_{t \rightarrow \infty} \int_0^\infty \bar{i}(t, a) da = 0,$$

which implies that

$$\lim_{t \rightarrow \infty} \int_0^\infty i(t, a) da = 0.$$

Next, we show $\lim_{t \rightarrow \infty} x(t) = 0$. Choose $\varepsilon \in (0, r)$. Since $\beta \in C_{BU}(\mathbb{R}_+, \mathbb{R}_+)$, $\lim_{t \rightarrow \infty} \int_0^\infty i(t, a) da = 0$ implies that $\lim_{t \rightarrow \infty} \int_0^\infty \beta(a) i(t, a) da = 0$. Then there exists a $t_0 \in \mathbb{R}_+$ such that $\int_0^\infty \beta(a) i(t, a) da \leq \varepsilon$ for $t \geq t_0$. By the third equation of (2.3), we get

$$\frac{dx}{dt} \leq \delta x(1 - x)(\varepsilon - r) \quad \text{for } t \geq t_0.$$

Recalling that $x(t) \in [0, 1]$ for $t \in \mathbb{R}_+$ and $\varepsilon < r$, we easily see that $\lim_{t \rightarrow \infty} x(t) = 0$.

Finally, we show $\lim_{t \rightarrow \infty} S(t) = \frac{\Lambda}{\mu}$. By Lemma 4.1, choose $\{s_n\}$ such that $s_n \rightarrow \infty$, $S(s_n) \rightarrow S_\infty$, and $\frac{dS(s_n)}{dt} \rightarrow 0$ as $n \rightarrow \infty$. Taking limit in

$$\frac{dS(s_n)}{dt} = \Lambda(1 - x(s_n)) - S(s_n) \int_0^\infty \beta(a) i(s_n, a) da - \mu S(s_n)$$

produces

$$0 = \Lambda - \mu S_\infty$$

or $S_\infty = \frac{\Lambda}{\mu}$. This, combined with $S^\infty \leq \frac{\Lambda}{\mu}$, yields $\lim_{t \rightarrow \infty} S(t) = \frac{\Lambda}{\mu}$.

To summarize, we have shown that $(S(t), i(t, a), x(t))$ converges to E_2 . This completes the proof. \square

For the attractivity of E_3 , we define

$$\Gamma_0 = \{(S, i, x) \in \Gamma : \int_0^{\bar{a}} i(a) da > 0, x \in [0, 1]\},$$

where $\bar{a} = \sup\{a \in \mathbb{R}_+ : \beta(a) > 0\}$. With the integrated semigroup approach, one can show (similarly as in Magal *et al.* [27], for example) that Γ_0 is a positively invariant subset for (2.3).

Theorem 4.3. *Suppose that $1 < \mathcal{R}_0 < 1 + \frac{r}{\mu}$. Then the equilibrium E_3 attracts all solutions of (2.3) with $(S_0, i_0, x_0) \in \Gamma_0$.*

Proof. Let $(S(t), i(t, a), x(t))$ be a solution of (2.3) with $(S_0, i_0, x_0) \in \Gamma_0$.

We first show $\lim_{t \rightarrow \infty} x(t) = 0$. From the proof of Theorem 4.2, we know that

$$0 \leq S(t) \leq \bar{S}(t) \quad \text{and} \quad 0 \leq i(t, a) \leq \bar{i}(t, a) \quad (4.2)$$

for $(t, a) \in \mathbb{R}_+^2$. Since $\mathcal{R}_0 > 1$, it follows from Theorem 2.3 that

$$\bar{S}(t) \rightarrow S^* \quad \text{and} \quad \bar{i}(t, \cdot) \rightarrow i_1^*(0)\pi(\cdot) \text{ in } L_+^1 \text{ as } t \rightarrow \infty. \quad (4.3)$$

Then we can obtain

$$\begin{aligned} \limsup_{t \rightarrow \infty} \int_0^\infty \beta(a) i(t, a) da &\leq \limsup_{t \rightarrow \infty} \int_0^\infty \beta(a) \bar{i}(t, a) da \\ &\leq \limsup_{t \rightarrow \infty} \left[\int_0^\infty \beta(a) i_1^*(0) \pi(a) da + \int_0^\infty \beta(a) |\bar{i}(t, a) - i_1^*(0) \pi(a)| da \right] \\ &= \mu(\mathcal{R}_0 - 1). \end{aligned}$$

Choose $\varepsilon > 0$ such that $\mu(\mathcal{R}_0 - 1) + \varepsilon < r$, which is possible since $\mathcal{R}_0 < 1 + \frac{r}{\mu}$. So there exists $t_0 \in \mathbb{R}_+$ such that

$$\int_0^\infty \beta(a) i(t, a) da \leq \mu(\mathcal{R}_0 - 1) + \varepsilon \quad \text{for } t \geq t_0.$$

By the third equation of (2.3), we have

$$\frac{dx(t)}{dt} \leq \delta x(t)(1 - x(t))(\mu(\mathcal{R}_0 - 1) + \varepsilon) \quad \text{for } t \geq t_0.$$

This, together with $\mu(\mathcal{R}_0 - 1) + \varepsilon - r < 0$ and $x(t) \in [0, 1]$ for $t \in \mathbb{R}_+$, gives $\lim_{t \rightarrow \infty} x(t) = 0$.

Next, we show that $S(t) \rightarrow S^*$ and $i(t, \cdot) \rightarrow i_1^*(0)\pi(\cdot)$ in L_+^1 as $t \rightarrow \infty$. For any $\eta \in (0, 1 - \frac{1}{\mathcal{R}_0})$, it follows from $\lim_{t \rightarrow \infty} x(t) = 0$ that there exists $t_1 \in \mathbb{R}_+$ such that

$$0 \leq x(t) \leq \eta \quad \text{for } t \geq t_1.$$

Then, for $t \geq t_1$, $(S(t), i(t, a))$ satisfies

$$\begin{cases} \frac{dS(t)}{dt} \geq \Lambda(1 - \eta) - S(t) \int_0^\infty \beta(a) i(t, a) da - \mu S(t), \\ \frac{\partial i(t, a)}{\partial t} + \frac{\partial i(t, a)}{\partial a} = -(mu + \gamma(a)) i(t, a). \end{cases}$$

Let $(\hat{S}(t), \hat{i}(t, a))$ be the solution of the auxiliary system

$$\begin{cases} \frac{d\hat{S}(t)}{dt} = \Lambda(1 - \eta) - \hat{S}(t) \int_0^\infty \beta(a) \hat{i}(t, a) da - \mu \hat{S}(t) \\ \frac{\partial \hat{i}(t, a)}{\partial t} + \frac{\partial \hat{i}(t, a)}{\partial a} = -(mu + \gamma(a)) \hat{i}(t, a) \end{cases}$$

with $(\hat{S}_0, \hat{i}_0) = (S(t_1), i(t_1, \cdot))$ and $\hat{i}(t, 0) = \hat{S}(t) \int_0^\infty \hat{i}(t, a) da$. Then applying the comparison principle yields

$$0 \leq \hat{S}(t) \leq S(t + t_1) \quad \text{and} \quad 0 \leq \hat{i}(t, a) \leq i(t + t_1, a) \quad (4.4)$$

for $(t, a) \in \mathbb{R}_+^2$. Since $(1 - \eta)\mathcal{R}_0 > 1$, applying Theorem 2.3 again, we see that

$$\hat{S}(t) \rightarrow S^* \quad \text{and} \quad \hat{i}(t, \cdot) \rightarrow \frac{\mu}{K}(\mathcal{R}_0(1 - \eta) - 1)\pi(\cdot) \text{ in } L_1^+ \text{ as } t \rightarrow \infty. \quad (4.5)$$

Notice that, for $(t, a) \in \mathbb{R}_+^2$,

$$\begin{aligned} |i(t, a) - i_1^*(0)\pi(a)| &\leq |\bar{i}(t, a) - i_1^*(0)\pi(a)| + |\hat{i}(t, a) - i_1^*(0)\pi(a)| \\ &\leq |\bar{i}(t, a) - i_1^*(0)\pi(a)| + |\hat{i}(t, a) - \frac{\mu}{K}(\mathcal{R}_0(1 - \eta) - 1)\pi(a)| + \Lambda\eta\pi(a). \end{aligned}$$

By (4.2)–(4.5), we have

$$S^* = \hat{S}_\infty \leq S_\infty \leq S^\infty \leq \bar{S}^\infty = S^*$$

and

$$\begin{aligned} \limsup_{t \rightarrow \infty} \int_0^\infty i(t, a) da &\leq \limsup_{t \rightarrow \infty} \left[\int_0^\infty |\bar{i}(t, a) - i_1^*(0)\pi(a)| da \right. \\ &\quad \left. + \int_0^\infty \left| \hat{i}(t, a) - \frac{\mu}{K}(\mathcal{R}_0(1 - \eta) - 1)\pi(a) \right| da + \int_0^\infty \Lambda\eta\pi(a) da \right] \\ &\leq \frac{\Lambda}{\mu}\eta. \end{aligned}$$

By the arbitrariness of η , we have $\limsup_{t \rightarrow \infty} \int_0^\infty |i(t, a) - i_1^*(0)\pi(a)| da = 0$. Therefore, we have proved that $S(t) \rightarrow S^*$ and $i(t, \cdot) \rightarrow i_1^*(0)\pi(\cdot)$ in L_1^+ as $t \rightarrow \infty$. This completes the proof. \square

Finally, we study the disease persistence of (2.3). Define $\rho : \Gamma \ni (S, i, x) \mapsto \int_0^\infty \beta(a) i(a) da$, which is the same as $\int_0^{\bar{a}} \beta(a) i(a) da$. Let

$$D_0 = \{S_0, i_0, x_0\} \in \Gamma : \text{there exists a } t \in \mathbb{R}_+ \text{ such that } \rho(\Phi(t, (S_0, i_0, x_0))) > 0 \text{ and } x \in [0, 1]\}.$$

We distinguish two kinds of persistence.

- The disease in (2.3) is **uniformly weakly ρ -persistent** if there exists an $\eta > 0$, independent of the initial conditions, such that if $(S_0, i_0, x_0) \in D_0$ then

$$\limsup_{t \rightarrow \infty} \rho(\Phi(t, (S_0, i_0, x_0))) > \eta.$$

- The disease in (2.3) is **uniformly strongly ρ -persistent** if there exists an $\eta > 0$, independent of the initial conditions, such that if $(S_0, i_0, x_0) \in D_0$ then

$$\liminf_{t \rightarrow \infty} \rho(\Phi(t, (S_0, i_0, x_0))) > \eta.$$

Lemma 4.4. Assume $\mathcal{R}_0 > 1 + \frac{r}{\mu}$. Then (2.3) is uniformly weakly ρ -persistent.

Proof. Since $\mathcal{R}_0 > 1 + \frac{r}{\mu}$, there exists an ϵ such that $0 < \epsilon < \min(1, r)$ and

$$\left(\frac{\Lambda(1-\epsilon)}{\mu+\epsilon} - \epsilon \right) \hat{K}(\epsilon) > 1. \quad (4.6)$$

By way of contradiction, we assume that there exists $(S_0, i_0, x_0) \in D_0$ such that

$$\limsup_{t \rightarrow \infty} \rho(\Phi(t, (S_0, i_0, x_0))) < \frac{\epsilon}{2}.$$

Then there is a $t_1 \in \mathbb{R}_+$ such that

$$\rho(\Phi(t, (S_0, i_0, x_0))) < \epsilon \quad \text{for } t \geq t_1.$$

As before, it follows from $x(t) \in [0, 1)$ for $t \in \mathbb{R}_+$, $\epsilon < r$, and

$$\frac{dx}{dt} \leq \delta x(1-x)(\epsilon-r) \quad \text{for } t \geq t_1$$

that $\lim_{t \rightarrow \infty} x(t) = 0$. Therefore, there is a $t_2 \geq t_1$ such that

$$0 \leq x(t) < \epsilon \quad \text{for } t \geq t_2.$$

As a result, we have

$$\frac{dS(t)}{dt} \geq \Lambda(1-\epsilon) - (\mu+\epsilon)S(t) \quad \text{for } t \geq t_2,$$

which implies that

$$\liminf_{t \rightarrow \infty} S(t) \geq \frac{\Lambda(1-\epsilon)}{\mu+\epsilon}.$$

Then there exists $t_3 \geq t_2$ such that

$$S(t) \geq \frac{\Lambda(1-\epsilon)}{\mu+\epsilon} - \epsilon \quad \text{for } t \geq t_3.$$

With the help of (2.5), we easily see that

$$B(t) \geq \left(\frac{\Lambda(1-\epsilon)}{\mu+\epsilon} - \epsilon \right) \int_0^t \beta(a)B(t-a)\pi(a)da \quad \text{for } t \geq t_3. \quad (4.7)$$

By replacing the initial condition with $(S(t_3), i(t_3, \cdot), x(t_3))$, we can assume that (4.7) holds for all $t \in \mathbb{R}_+$. Note that both $B(\cdot)$ and $\beta(\cdot)\pi(\cdot)$ are bounded functions on \mathbb{R}_+ and hence their Laplace transforms exist at least on $(0, \infty)$. It follows from (4.7) (with $t_3 = 0$) that

$$\hat{B}(\lambda) \geq \left(\frac{\Lambda(1-\epsilon)}{\mu+\epsilon} - \epsilon \right) \hat{B}(\lambda) \hat{K}(\lambda) \quad \text{for } \lambda > 0, \quad (4.8)$$

where $\hat{\cdot}$ denotes the Laplace transform of a function. As $B(\cdot)$ is not identically zero on \mathbb{R}_+ , we know that $\hat{B}(\lambda) > 0$ for $\lambda \in (0, \infty)$. It follows from (4.8) that $(\frac{\Lambda(1-\epsilon)}{\mu+\epsilon} - \epsilon) \hat{K}(\lambda) \leq 1$ for $\lambda \in (0, \infty)$. In particular, $(\frac{\Lambda(1-\epsilon)}{\mu+\epsilon} - \epsilon) \hat{K}(\epsilon) \leq 1$, which contradicts with (4.6). This completes the proof. \square

Next, we establish the uniform strong ρ -persistence. For this purpose, it is crucial to show Φ has a global compact attractor in D_0 . A global compact attractor \mathcal{A} is a maximal compact invariant set in D_0 such that for any open set that contains \mathcal{A} , all solutions of (2.3) that start at zero from a bounded set, are contained in that open set, at least for sufficiently large time. To establish the existence of global attractors, we need the following three results.

Lemma 4.5 ([30, Theorem 3.4.6]). *If $T(t) : X \rightarrow X$, $t \in \mathbb{R}_+$ is asymptotically smooth, point dissipative and orbits of bounded sets are bounded, then there exists a global attractor.*

A semiflow is called *asymptotically smooth* if each forward invariant bounded closed set is attracted by a nonempty compact set.

Lemma 4.6 ([30, Lemma 3.2.3]). *For each $t \in \mathbb{R}_+$, suppose $T(t) = S(t) + U(t) : X \rightarrow X$ has the property that $U(t)$ is complete continuous and there is a continuous function $k : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $k(t, r) \rightarrow 0$ as $t \rightarrow \infty$ and $|S(t)x| \leq k(t, r)$ if $|x| < r$. Then $T(t)$, $t \in \mathbb{R}_+$, is asymptotically smooth.*

The next result will be need in the discussion of Φ being asymptotically smooth.

Lemma 4.7. *For any $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$|B(t+h) - B(t)| \leq \varepsilon \quad \text{for all } t \in \mathbb{R}_+, 0 < h < \delta, \text{ and } (S_0, i_0, x_0) \in \Gamma. \quad (4.9)$$

Proof. It is easy to see that $B(t) \leq \bar{\beta}(\frac{\Lambda}{\mu})^2$ for $t \in \mathbb{R}_+$. Now, let $t \in \mathbb{R}_+$ and $h > 0$. Then

$$\begin{aligned} |B(t+h) - B(t)| &= \left| S(t+h) \int_0^\infty \beta(a)i(t+h, a)da - S(t) \int_0^\infty \beta(a)i(t, a)da \right| \\ &\leq \left| [S(t+h) - S(t)] \int_0^\infty \beta(a)i(t+h, a)da \right| \\ &\quad + \left| S(t) \left[\int_0^\infty \beta(a)i(t+h, a)da - \int_0^\infty \beta(a)i(t, a)da \right] \right| \\ &\leq \left(2\Lambda + \bar{\beta} \left(\frac{\Lambda}{\mu} \right)^2 \right) h \bar{\beta} \frac{\Lambda}{\mu} + \frac{\Lambda}{\mu} \int_0^h \beta(a)i(t+h, a)da \\ &\quad + \frac{\Lambda}{\mu} \left| \int_h^\infty \beta(a)i(t+h, a)da - \int_0^\infty \beta(a)i(t, a)da \right| \\ &= \left(2\Lambda + \bar{\beta} \left(\frac{\Lambda}{\mu} \right)^2 \right) h \bar{\beta} \frac{\Lambda}{\mu} + \frac{\Lambda}{\mu} \int_0^h \beta(a)B(t+h-a)\pi(a)da \\ &\quad + \frac{\Lambda}{\mu} \left| \int_h^\infty \beta(a)i(t+h, a)da - \int_0^\infty \beta(a)i(t, a)da \right| \\ &\leq \left(2\Lambda + \bar{\beta} \left(\frac{\Lambda}{\mu} \right)^2 \right) h \bar{\beta} \frac{\Lambda}{\mu} + \frac{\Lambda}{\mu} \bar{\beta} \bar{\beta} \left(\frac{\Lambda}{\mu} \right)^2 h \\ &\quad + \frac{\Lambda}{\mu} \left| \int_h^\infty \beta(a)i(t+h, a)da - \int_0^\infty \beta(a)i(t, a)da \right|, \end{aligned}$$

or

$$|B(t+h) - B(t)| \leq 2\bar{\beta} \frac{\Lambda}{\mu} \left[\Lambda + \left(\frac{\Lambda}{\mu} \right)^2 \right] h + \frac{\Lambda}{\mu} \left| \int_h^\infty \beta(a)i(t+h, a)da - \int_0^\infty \beta(a)i(t, a)da \right|. \quad (4.10)$$

We now come to estimate

$$\left| \int_h^\infty \beta(a)i(t+h, a)da - \int_0^\infty \beta(a)i(t, a)da \right| = \left| \int_0^\infty [\beta(a+h)i(t+h, a+h) - \beta(a)i(t, a)]da \right|.$$

Note that, by (2.5), we have

$$i(t+h, a+h) = i(t, a) \frac{\pi(a+h)}{\pi(a)} \quad \text{for all } (a, t, h) \in \mathbb{R}_+^3.$$

Therefore,

$$\begin{aligned} & \left| \int_h^\infty \beta(a)i(t+h, a)da - \int_0^\infty \beta(a)i(t, a)da \right| \\ &= \left| \int_0^\infty \left(\beta(a+h) \frac{\pi(a+h)}{\pi(a)} - \beta(a) \right) i(t, a)da \right| \\ &\leq \int_0^\infty \beta(a+h) \left(1 - e^{-\int_a^{a+h} k(s)ds} \right) i(t, a)da + \int_0^\infty |\beta(a+h) - \beta(a)| i(t, a)da. \end{aligned}$$

Since $-\bar{k}h \leq -\int_a^{a+h} k(s)da \leq 0$ and $e^{-x} \geq 1 - x$ for $x \in \mathbb{R}_+$, we can get

$$\left| \int_h^\infty \beta(a)i(t+h, a)da - \int_0^\infty \beta(a)i(t, a)da \right| \leq \bar{\beta}\bar{k}h \frac{\Lambda}{\mu} + \int_0^\infty |\beta(a+h) - \beta(a)| i(t, a)da.$$

This, together with (4.10) and the fact that β is uniformly continuous, immediately produces (4.9). \square

By Lemma 4.4, we know that $\Phi(t, D_0) \subseteq D_0$ for $t \in \mathbb{R}_+$. So it induces a semiflow on D_0 .

Lemma 4.8. *If $\mathcal{R}_0 > 1 + \frac{r}{\mu}$, then there exists a global attractor for the solution semiflow Φ of (2.3) in D_0 .*

Proof. With the help of Proposition 2.2 and Lemma 4.5, we only need to show that the restricted semiflow on D_0 is asymptotically smooth. This is done by applying Lemma 4.6 as follows.

For $t \in \mathbb{R}_+$ and $(S_0, i_0, x_0) \in D_0$, let

$$\begin{aligned} \hat{\Phi}(t, (S_0, i_0, x_0)) &= (0, \hat{i}(t, \cdot), 0), \\ \tilde{\Phi}(t, (S_0, i_0, x_0)) &= (S(t), \tilde{i}(t, \cdot), x(t)), \end{aligned}$$

where

$$\tilde{i}(t, a) = \begin{cases} i(t, a) & \text{for } 0 \leq a \leq t \\ 0 & \text{for } t < a \end{cases} = \begin{cases} B(t-a)\pi(a) & \text{for } 0 \leq a \leq t \\ 0 & \text{for } t < a \end{cases} \quad (4.11)$$

and

$$\hat{i}(t, a) = i(t, a) - \tilde{i}(t, a) = \begin{cases} 0 & \text{for } 0 \leq a \leq t, \\ i_0(a-t) \frac{\pi(a)}{\pi(a-t)} & \text{for } t < a. \end{cases} \quad (4.12)$$

Then $\Phi = \hat{\Phi} + \tilde{\Phi}$. Clearly, both \hat{i} and \tilde{i} are nonnegative. It follows from (4.12) that

$$\begin{aligned} \|\hat{\Phi}(t, (S_0, i_0, x_0))\| &= \|\hat{i}(t, \cdot)\|_1 \\ &= \int_t^\infty i_0(a-t) \frac{\pi(a)}{\pi(a-t)} da \\ &= \int_0^\infty i_0(a) \frac{\pi(a+t)}{\pi(a)} da \\ &\leq e^{-\mu t} \int_0^\infty i_0(a) da \\ &= e^{-\mu t} \|i_0\|_1 \\ &\leq e^{-\mu t} \|(S_0, i_0, x_0)\| \end{aligned}$$

and hence $\hat{\Phi}$ satisfies the assumption in Lemma 4.6.

Now, we show that $\tilde{\Phi}$ is completely continuous, that is, the set $\{\tilde{\Phi}(t, (S_0, i_0, x_0)) : (S_0, i_0, x_0) \in \mathbb{B}\}$ is precompact for any fixed $t \in \mathbb{R}_+$ and any bounded set $\mathbb{B} \subseteq D_0$. This is done by applying the Fréchet-Kolmogorov Theorem [31]. First, it follows easily from the definitions of $\tilde{\Phi}$, D_0 , and Γ that $\{\tilde{\Phi}(t, (S_0, i_0, x_0)) : (S_0, i_0, x_0) \in \mathbb{B}\}$ is bounded and this verifies the first condition of the Fréchet-Kolmogorov Theorem. Second, by (4.11), the third condition of Fréchet-Kolmogorov Theorem is satisfied. Finally, we verify the second condition of the Fréchet-Kolmogorov Theorem. It suffices to show that

$$\lim_{h \rightarrow 0^+} \|\tilde{i}(t, \cdot) - \tilde{i}(t, \cdot + h)\|_1 = 0 \quad \text{uniformly in } \tilde{i}(t, \cdot) \in \{\tilde{\Phi}(t, (S_0, i_0, x_0)) : (S_0, i_0, x_0) \in \mathbb{B}\}. \quad (4.13)$$

If $t = 0$ then (4.13) holds automatically since $\tilde{i}(0, \cdot) = 0$ by (4.11). Without loss of generality, we assume that $t > 0$. Since we concern with the limit as h tends to 0^+ , we only consider $h \in (0, t)$. Then

$$\begin{aligned} \|\tilde{i}(t, \cdot) - \tilde{i}(t, \cdot + h)\|_1 &= \int_0^\infty |\tilde{i}(t, a) - \tilde{i}(t, a + h)| da \\ &= \int_0^{t-h} |B(t-a-h)\pi(a+h) - B(t-a)\pi(a)| da + \int_{t-h}^t B(t-a)\pi(a) da \\ &\leq \bar{\beta} \left(\frac{\Lambda}{\mu}\right)^2 h + \int_0^{t-h} |B(t-a-h)\pi(a+h) - B(t-a)\pi(a)| da \\ &\leq \bar{\beta} \left(\frac{\Lambda}{\mu}\right)^2 h + \int_0^{t-h} B(t-a-h) |\pi(a+h) - \pi(a)| da \\ &\quad + \int_0^{t-h} |B(t-a-h) - B(t-a)| \pi(a) da \\ &\leq \bar{\beta} \left(\frac{\Lambda}{\mu}\right)^2 h(1 + t\bar{k}) + \int_0^{t-h} |B(t-a-h) - B(t-a)| \pi(a) da \end{aligned}$$

as we know $B(t) \leq \bar{\beta}(\frac{\Lambda}{\mu})^2$ for $t \in \mathbb{R}_+$ and $|\pi(a+h) - \pi(a)| = \pi(a) \left[1 - e^{-\int_a^{a+h} k(s) ds}\right] \leq \bar{k}h$. This, combined with Lemma 4.7, immediately gives (4.13) and hence we have completed the proof. \square

Now, with the assistance of Lemma 4.4, Lemma 4.8, and [32, Theorem 2.3], we can obtain the following result.

Theorem 4.9. *If $\mathcal{R}_0 > 1 + \frac{\tau}{\mu}$, then (2.3) is uniformly strongly ρ -persistent.*

5 Numerical simulations

In this section, we always assume that Assumption 3.1 holds. We provide some simulations to illustrate the theoretical results obtained in the previous sections. Here $I(t) = \int_0^\infty i(t, a) da$ for $t \in \mathbb{R}_+$.

First, set $\Lambda = 0.4$, $r = 0.9$, $\delta = 0.8$, $\tau = 1.25$, $\beta = 0.001$, $\mu = 0.01$, and $\gamma = 0.1$. Then $\mathcal{R}_0 = 0.3591 < 1$. By Theorem 4.2, E_2 attracts all solutions with initial conditions in $\{(S, i, x) \in \Gamma : x \in [0, 1)\}$. Fig. 1(a) supports this with the initial condition $(S_0, i_0, x_0) = (0.5, 50e^{-.4a}, 0.1)$. In Fig. 1(b), we plot the phase diagram in the SI -plane with five different initial conditions.

Second, enlarge the transmission rate to $\beta = 0.02$ while keeping the other parameters at the same values as in the first case. Then $\mathcal{R}_0 = 7.1824$, which is between 1 and $1 + \frac{\tau}{\mu} = 91$. It follows

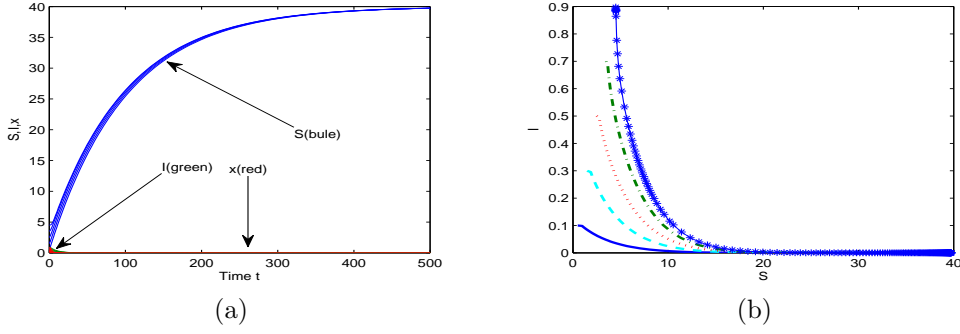


Figure 1: With $\Lambda = 0.4$, $r = 0.9$, $\delta = 0.8$, $\tau = 1.25$, $\beta = 0.001$, $\mu = 0.01$, and $\gamma = 0.1$, the equilibrium E_2 attracts all solutions with initial conditions in $\{(S, i, x) \in \Gamma : x \in [0, 1]\}$. (a) Time series of S , I , and x with five different initial conditions; (b) The phase diagram in the SI -plane with five different initial conditions.

from Theorem 3.2 and Theorem 4.3 that the endemic and vaccinator equilibrium E_3 is globally asymptotically stable. Figure 2 indicates that (2.3) evolves towards E_3 .

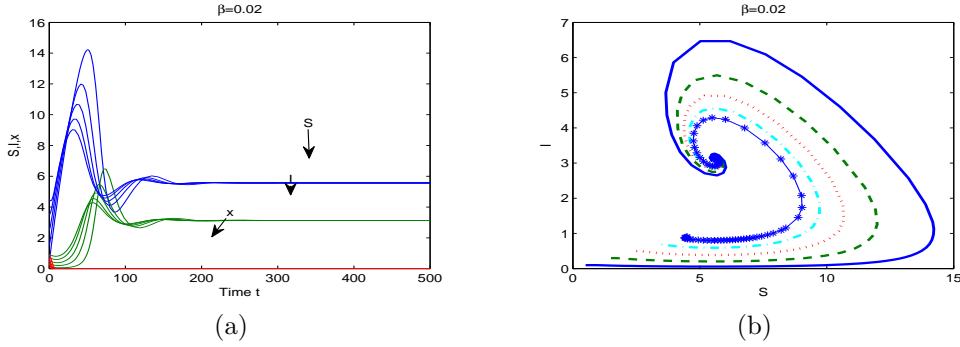


Figure 2: With $\Lambda = 0.4$, $r = 0.9$, $\delta = 0.8$, $\tau = 1.25$, $\beta = 0.02$, $\mu = 0.01$, and $\gamma = 0.1$, the equilibrium E_3 is locally asymptotically stable. (a) Times series of S (blue), I (green), and x (red) with five different initial conditions, (b) the corresponding phase diagram in the SI -plane.

Suppose the cost for the vaccination is too small and let $r = 0.000001$. From the third equation of (2.3) we know that x monotonously increases to 1. Then (2.3) has a pure vaccinator equilibrium E_1 , which is unstable by Theorem 3.2 (see Figure 3(a)). If the cost of the vaccination is free or low, then the coverage of the proportion vaccinated is towards high values. However, persons have no incentives to vaccinate and the pure vaccinator equilibrium is unstable.

We rise the cost for vaccination to $r = 0.01$ (which is between 0.000001 (lower cost) and 1.2 (higher cost)) and keep the other parameters unchanged as in the second case. Note that $\delta = 0.5 < 1 < \frac{\beta\Lambda}{\beta\Lambda - (\mu + \gamma)(\mu + r)}$. Theorem 3.2 implies that the endemic and vaccinator equilibrium E_4 is locally asymptotically stable, which is supported in Figure 3(b).

Theorem 3.2, Proposition 3.3, and Theorem 3.4 imply that δ , τ , and r play an important role in the evolution of (2.3). Recall that δ represents the imitation rate describing the imitation behaviors and the vaccination behavior for children. Let $\Lambda = 0.4$, $r = 0.01$, $\tau = 1.25$, $\beta = 0.05$, $\mu = \frac{1}{50 \times 365}$,

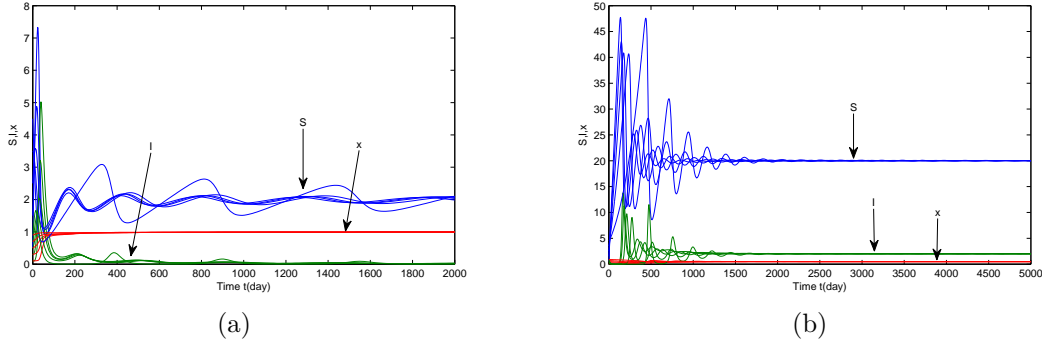


Figure 3: Representative time series of S (blue), I (green), and x (red). (a) E_1 is unstable, here $\Lambda = 0.4$, $r = 0.000001$, $\delta = 0.5$, $\tau = 1.25$, $\beta = 0.005$, $\mu = \frac{1}{50 \times 365}$, and $\gamma = 0.1$; (b) E_4 is locally asymptotically stable with $\Lambda = 0.4$, $r = 0.01$, $\delta = 0.5$, $\tau = 1.25$, $\beta = 0.005$, $\mu = \frac{1}{50 \times 365}$, and $\gamma = 0.1$.

and $\gamma = 0.1$. For some larger imitation rate δ , system (2.3) exhibits limit cycles and hence it can be destabilized by the imitation rate δ . Fig. 4(a) shows the bifurcation diagram by taking δ as a

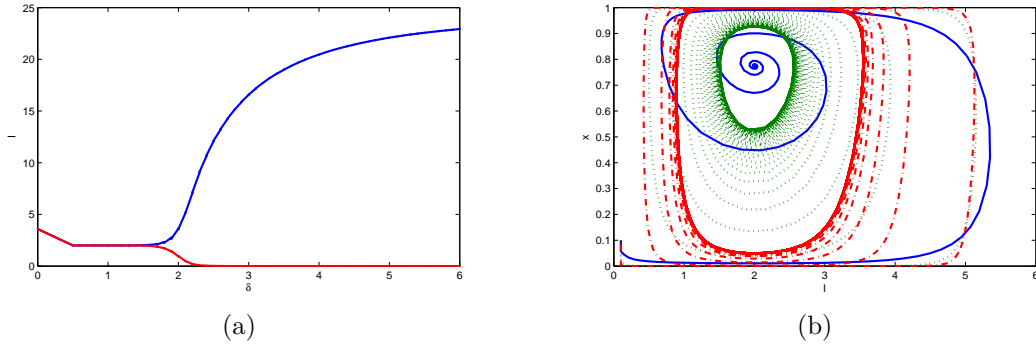


Figure 4: With $\Lambda = 0.4$, $r = 0.01$, $\tau = 1.25$, $\beta = 0.05$, $\mu = \frac{1}{50 \times 365}$, and $\gamma = 0.1$, the imitation rate δ can destabilize (2.3). (a) The bifurcation diagram showing I with respect to the imitation rate δ with δ varying from 0 to 6; (b) Representative phase diagram in the Ix -plane with δ being 0.5 (blue), 1.5 (green), and 2.5 (red), respectively and the same initial condition $(S_0, i_0, x_0) = (0.5, 5e^{-0.4a}, 0.1)$

bifurcation parameter, which indicates that the amplitude of the oscillations increases as the imitation rate does; while Fig. 4(b) shows the phase diagram in the Ix -plane with $\delta = 0.5, 1.5$, and 2.5 , and the initial condition $(S_0, i_0, x_0) = (0.5, 5e^{-0.4a}, 0.1)$.

The perceived vaccine risk r impacts the prevalence of the disease spread. If the cost of the vaccination is high, such as a vaccine scare or a strong side effect, the frequency of the vaccination will gradually decrease and the prevalence of the disease will increase. Fig. 5(a) shows the infection equilibrium structure of (2.3) in terms of the disease prevalence vs r with $\Lambda = 0.4$, $\delta = 0.5$, $\tau = 1.25$, $\beta = 0.05$, $\mu = \frac{1}{50 \times 365}$, and $\gamma = 0.1$. We can see that the disease prevalence increases when the cost of vaccination increases. Moreover, the equilibrium E_4 changes stability with r in the interval $[0, 0.2]$ at $r = 0.01$ from stable to unstable.

The latent period τ is also a key parameter for controlling the disease. For a childhood disease,

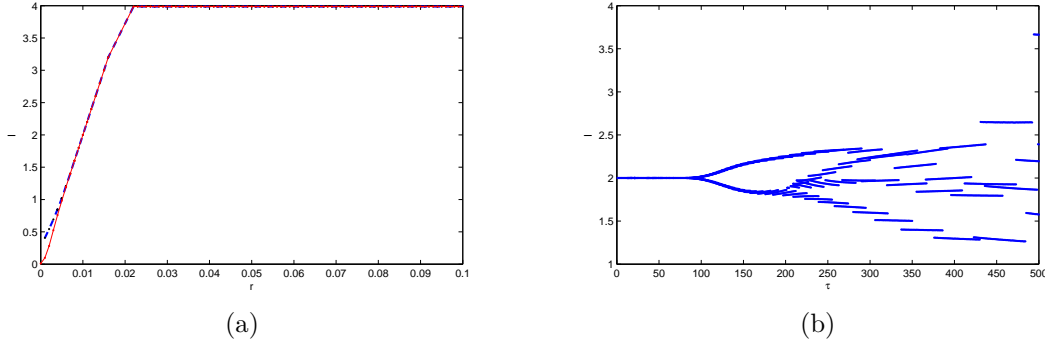


Figure 5: With $\Lambda = 0.4$, $\delta = 0.5$, $\beta = 0.05$, $\mu = \frac{1}{50 \times 365}$, and $\gamma = 0.1$, (a) infection equilibrium structure of (2.3) with $\tau = 1.25$; (b) bifurcation diagram with the latent period τ as bifurcation parameter and $r = 0.01$.

if some medicine can extend the latent period of the disease, the prevalence of the disease can be lowered. Figure 5(b) shows that τ can destabilize (2.3) and oscillations occur when τ is from 100 to 500. Here $r = 0.01$ and the other parameters except τ have the same values as above for the impact of r . Also, the prevalence of the disease gradually increases with respect to τ .

6 Discussion

From the theoretical analysis in the above sections, the pure vaccinator equilibrium E_1 is always unstable. Parents have no incentive to vaccinate if the vaccination coverage is high. The disease-free and non-vaccinator equilibrium E_2 is globally asymptotically stable if and only if $\mathcal{R}_0 < 1$. If $1 < \mathcal{R}_0$, then there is an endemic and non-vaccinator equilibrium E_3 , which is locally asymptotically stable if $1 < \mathcal{R}_0 < 1 + \frac{\mu}{r}$. The endemic and vaccinator equilibrium E_4 is locally asymptotically stable if $\mathcal{R}_0 > 1 + \frac{\mu}{r}$, $\delta < \frac{\Lambda\beta}{\beta\Lambda - (\mu + \gamma)(\mu + r)}$ and $\mathbb{S} = \emptyset$. If $\mathbb{S} = \{\tau_0, \dots, \tau_{n_0}\}$ with $s_{n(\tau_0)}(\tau_0) > 0$, then a Hopf bifurcation occurs from the equilibrium E_4 , which implies that (2.3) can be destabilized.

The imitation rate δ is the main parameter that leads to destabilization of the system and to a Hopf bifurcation, which is proved in [14, 16]. The amplitude of the oscillations increases with the increase of the imitated behaviors. Rational decision-making individuals, depending on updated information about the perceived vaccine risk compared with the prevalence of the disease, decide whether or not to vaccinate their children. As mentioned earlier in Section 5, individuals imitate others more readily as the amount of information for the significant side effects of vaccination increases, which in turn enhances the difficulties in disease control. The difference in immunity, characteristic for childhood diseases, is incorporated through $\beta(a)$, the different transmission ability for the different values of the infection age a . For special cases, different transmissibility can be described by the parameters τ and β . Suitable higher value of τ will produce a limit cycle.

When there is no cost for vaccination, i.e., $r = 0$, the payoff benefit is larger than the cost of vaccination as long as $\int_0^\infty \beta(a)i(t, a)da$ is not zero. From the third equation of (2.3), the frequency of vaccination monotonously increases. It follows from the first two equations in combination with the

third equation of (2.3) that system (2.3) evolves to E_2 . On the other hand, if the cost of vaccination is not free, and the initial conditions satisfy $\int_0^\infty \beta(a)i(t_0, a)da > r$, the frequency of the vaccination monotonously increases. It follows from the first and second equation of (2.3) that infected individuals decrease until $\int_0^\infty \beta(a)i^*(a)da = r$. System (2.3) evolves to the endemic and non-vaccinator equilibrium E_3 or a stable endemic and vaccinator equilibrium E_4 which depends on whether the initial vaccinated proportion is equal or larger than 0. When $\int_0^\infty \beta(a)i(t, a)da < r$, it follows from the third equation of (2.3) that the proportion vaccinated x decreases. Prevalence of the disease increases until $\int_0^\infty \beta(a)i^*(a)da = r$. This implies that (2.3) also evolves to the endemic and non-vaccinator equilibrium E_3 or the endemic and vaccinator equilibrium E_4 which depends on the initial vaccinated proportion x_0 .

In summary, parents can make rational decisions in favor of disease control if they understand the interplay between the perceived vaccination risk, prevalence of the disease, and the variable transmission abilities of the children. Our investigation implies that high vaccination coverage can not guarantee the elimination of the disease for a voluntary vaccination policy. On the other hand, limited vaccination may be harmful for the control of the disease spread. The amount of up-to-date information for the vaccine use has two opposite effects: knowledge about the disease prevalence encourages parents to take the vaccine for their children, on the other hand the potential side effects discourage them to take vaccine for their children. Based on the game theory, we conclude that it is important that parents make rational decision whether or not to vaccinate their children.

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