2.3. Newton's Method

1. Newton's Method
2. The Secant Method
3. The Method of False Position

Newton's Method

- Problem: Given $f(x) = 0$. Find x in[a,b].
- Newton's Method is one of the most powerful and methods for solving root-finding problems.
- Newton's method is extremely fast, much faster than most iterative methods we can design.
- Newton's method is also called Newton-Raphson method

Derivation of Newton's Method from Taylor's Expansion.

- Suppose f(x) is twice continuously differentiable on [a,b].
- Let $f(p)=0$.
- Let $\eta \approx p$, so that
  - $|p - \eta|$ – small;
  - $f'(\eta) \neq 0$.
- Consider the Taylor's polynomial expansion of f(x) around η:

$$f(x) = f(\eta) + f'(\eta)(x-\eta)+f''(\xi(x))\,(x-\eta)^2/2!$$

Where $\xi(x)$ is a point between x and η.

Derivation of Newton's Method from Taylor's Expansion

- Set x=p and note that f(p)=0.

$0 = f(\eta)+f'(\eta)(p-\eta)+f''(\xi(x))(p-\eta)^2/2!$ (ignore!)

$0 \approx f(\eta)+f'(\eta)(p-\eta)$

Solving for p:

$$p \approx \eta - \frac{f(\eta)}{f'(\eta)}$$

- Newton's method: Given p_0 – initial guess of the root, the remaining approximations are computed from

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$$

Newton's Method

- Newton's method is a fixed point iteration method:

$$p_n = g(p_{n-1})$$

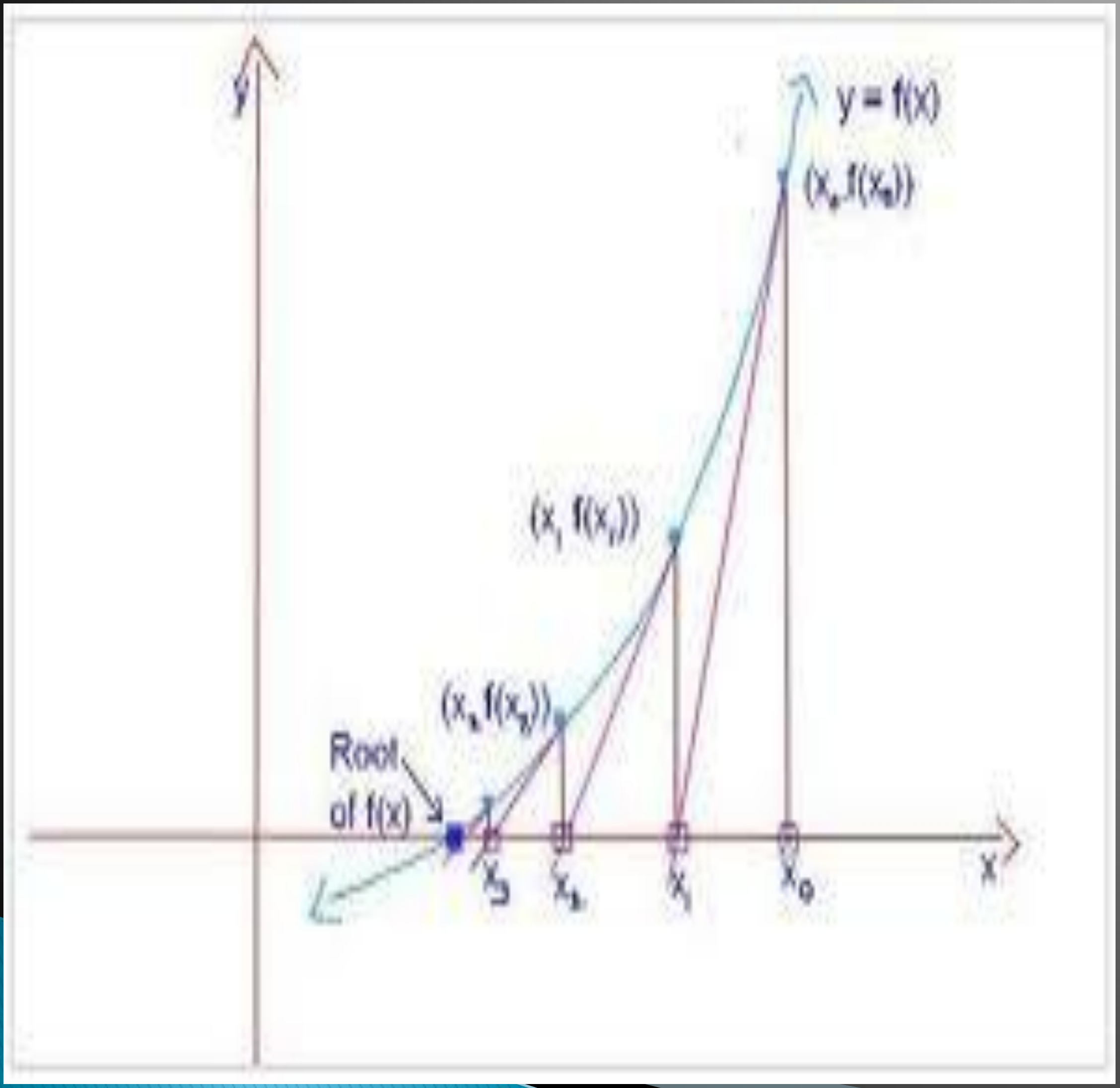
where

$$g(x) = x - \frac{f(x)}{f'(x)}$$

- Newton's method cannot continue if for some p_{n-1}

$$f'(p_{n-1}) = 0$$

Geometrical Interpretation of Newton's Method



- Choose p_0
- Draw the tangent at $$(p_0, f(p_0))$$
- This tangent crosses the x-axis at p_1
- Continue

Examples

- Use Newton's method to find the solution accurate to within 10^{-5} for the problem

  (x–2)^2 – ln x =0 for $1 \le x \le 2$

<u>Solution:</u> f(x) = (x–2)^2– ln x

f'(x) = 2(x–2)–1/x

The Newton's method becomes:

$$p_{n+1} = p_n - \frac{(p_n - 2)^2 - \ln p_n}{2(p_n - 2) - \frac{1}{p_n}}$$

Choose $p_0 = 1$ and run the iteration.

Examples

n	p_n	Error
0	1	
1	1.333333333	
2	1.408579272	
3	1.412381564	
4	1.412391172	0.96*10^{-5}

Comparing Newton's Method to Fixed-Point Iterations

- Rewrite the problem

$$(x-2)^2 - \ln x = 0$$

As a number of fixed point iterations. Compare their convergence with Newton's method.

Case A:

$$x = e^{(x-2)^2}$$

$$g(x) = e^{(x-2)^2}$$

Comparing Newton's Method to Fixed-Point Iterations

n	$g(p_n)$	n	$g(p_n)$
0	1.5	7	1.330077499
1	1.284025417	8	1.566425321
2	1.669659317	9	1.20681783
3	1.115301717	10	1.875992692
4	2.187350626	11	1.015496659
5	1.035723542	12	2.635958381
6	2.534076034		

- The iteration does not converge. Even after 100 iterations, it keeps on jumping from value to value

Theorem 2.3 does not hold. Conditions b) and d) fail.

Comparing Newton's Method to Fixed-Point Iterations

Case B:

$$(x-2)^2 = \ln x$$

$$x - 2 = \sqrt{\ln x}$$

$$x = 2 + \sqrt{\ln x}$$

$$g(x) = 2 + \sqrt{\ln x}$$

Comparing Newton's Method to Fixed-Point Iterations

n	$g(p_n)$
0	2
1	2.832554611
2	3.020381788
3	3.051372
4	3.0562156
5	3.0569661
6	3.0570823

- This is converging but to a root $p \approx 3.057$ which is not in the interval [1,2].
- Oops. What went wrong?
- We should have taken the negative root

$$x - 2 = -\sqrt{\ln x}$$

$$x = 2 - \sqrt{\ln x}$$

Iteration

This will converge to a root in the interval [1,2].

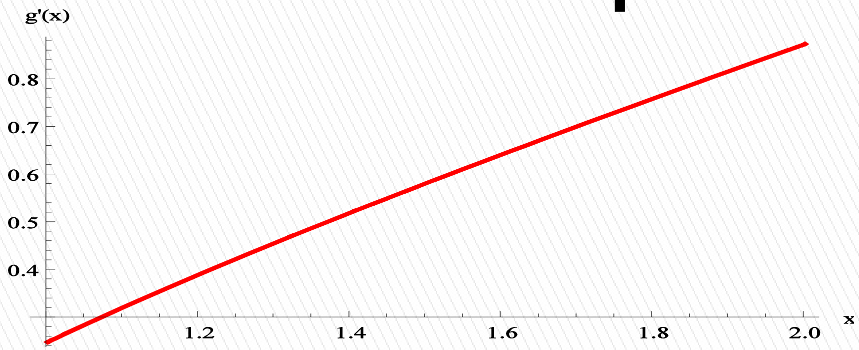
Comparing Newton's Method to Fixed-Point Iterations

- **Case C**:

$$x^2 - 4x + 4 = \ln x$$

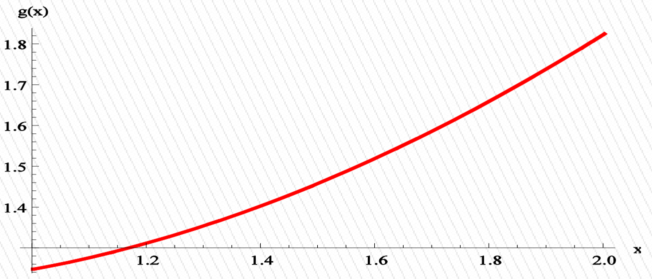
$$x = \frac{x^2 - \ln x + 4}{4}$$

$$g(x) = \frac{x^2 - \ln x + 4}{4}$$



- Conditions of Thm 2.3:

1. g(x) is continuous;



2. $1 < g(1) < g(x) < g(2) < 2$
3. g'(x) exists

$$g'(x) = \frac{x}{2} - \frac{1}{4x}$$

4. g'(x) is increasing and positive

$$|g'(x)| \le g'(2) = 7/8 = k < 1$$

Comparing Newton's Method to Fixed-Point Iterations

- Thm 2.3 applies so a fixed point iteration will converge for every p_0. The rate of convergence is

$$O\left(\frac{7}{8}^n\right)$$

- The relative error

$$\frac{|p_{16}-p_{15}|}{p_{16}} = 2.33*10^{-6}$$

n	p_n	n	p_n
0	1.5	9	1.412709914
1	1.461133723	10	1.412559879
2	1.438924775	11	1.412480459
3	1.426652089	12	1.412438425
4	1.42000142	13	1.412416178
5	1.41643654	14	1.412404405
6	1.414537058	15	1.412398175
7	1.413528195	16	1.412394878
8	1.412993278	17	1.412393133

Importance of the Choice of p_0

- **Theorem 2.5**. Let f(x) be twice continuously differentiable on [a,b]. If p is such that f(p)=0 and f'(p)≠0 then there exists a δ>0 such that Newton's method generates a sequence

  $p_0, p_1, \ldots, p_n, \ldots.$

  converging to p for any p_0 in the interval [p−δ,p+δ].
- This Thm says that if we start from p0 which is close enough to the root p, then Newton's method will converge.

Importance of the Choice of p_0

- **Example**: (showing the importance of p0 in Newton's method).
- Use Newton's method to find the solution of

$$x^3 - 6x^2 + 11x - 12 = 0$$

in the interval [2,5].

- **Solution**: The root is p=4. The Newton's method is given by:

$$p_{n+1} = p_n - \frac{p_n^3 - 6p_n^2 + 11p_n - 12}{3p_n^2 - 12p_n + 11}$$

Importance of the Choice of p_0

n	p_n	n	p_n
0	7/3	11	1.372802817
1	−7.1111111	12	32.57
2	−4.0743727	13	22.3894449
3	−2.03180127	14	15.60868858
4	−0.6185271	15	11.09963675
5	0.47170432	16	8.115195433
6	1.81034945	17	6.167426354
7	−4.71042586	18	4.95
8	−2.4622339	19	4.283998419
9	−0.92331673		…..
10	0.215552343	23	4.000000000

n	p_n
0	3
1	6
2	4.85106383
3	4.2385529
4	4.02626569
5	4.000368955
6	4.000000074
7	4.000000000

Converges in 23 iterations

Converges in 7 iterations

How Do We Locate p_0 close to p?

- Suppose the interval [a,b] is large.
- Suppose we don't know where the root is.
- How do we choose p_0 close enough to p?

- Run several steps of the bisection method to determine a smaller interval $[a_1,b_1]$ that contains the root.

- In the previous example 2 steps of the bisection method would have given the interval [3.5,4.25]. If we chose p0 to be the midpoint, we would have been very close to the root.

Secant Method

Idea of the Secant Method

- Weakness of Newton's method is that it needs f'(x) which may be difficult to find.
- Secant method computes an approximation of the solution of

$$f(x)=0$$

without the need of f'(x).

- The idea of the secant method is to substitute the slope of the tangent line, given by $f'(p_n)$ with the slope of the secant line through the points p_{n-1} and p_{n-2}.

Secant Method

- Recall Newton's method:

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$$

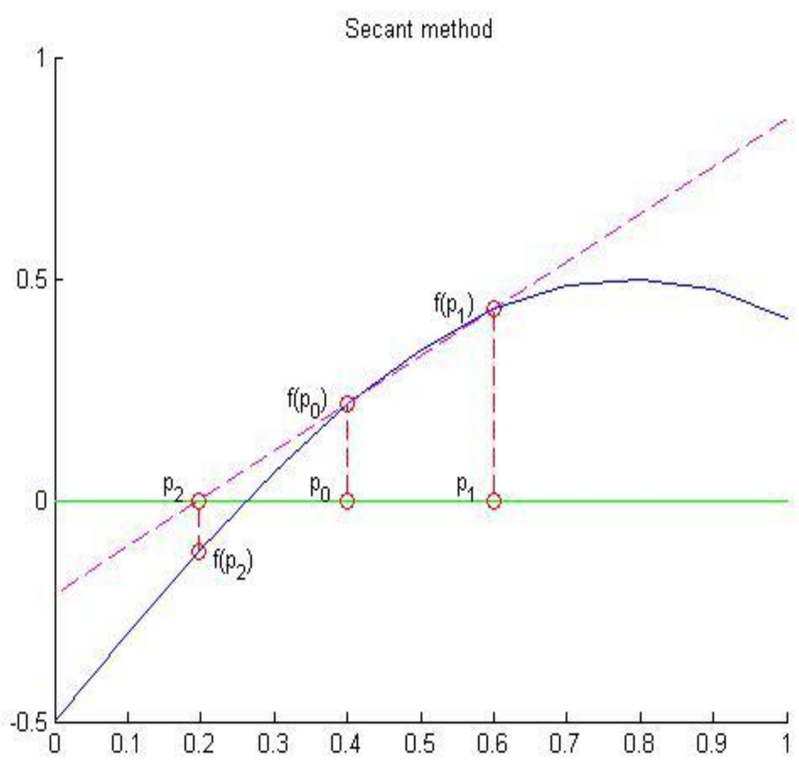
- The idea is to replace f'(x) with the slope of a secant line through p_{n-1} and p_{n-2}:

$$f'(p_{n-1}) \approx \frac{f(p_{n-1}) - f(p_{n-2})}{p_{n-1} - p_{n-2}}$$

- Replacing the derivative, we obtain the following formula for the Secant method:

$$p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}$$

Secant method: Geometrical Interpretation



Geometrical interpretation

- We need 2 initial values to start the iteration: p_0 and p_1.
- We draw the secant line through p_0 and p_1. We find p_2 from the intersection of the secant line with the x-axis.
- We draw a new secant line through p_1 and p_2.
- Continue.

Example for the Secant Method

- **<u>Example</u>**: Using secant method find the solution of the following equation in [1,2].

$$(x-2)^2 - \ln x = 0$$

- Let $p_0=1$ and $p_1=1.5$

n	pn
0	1
1	1.5
2	1.432726178
3	1.411129929
4	1.412408392
5	1.412391186
6	1.412391172

Secant method is a little slower than Newton’s method but faster than the bisection method and most fixed-point iterations.

Newton’s method arrived at the value 1.412391172 in 4 iterations.

Method of False Position

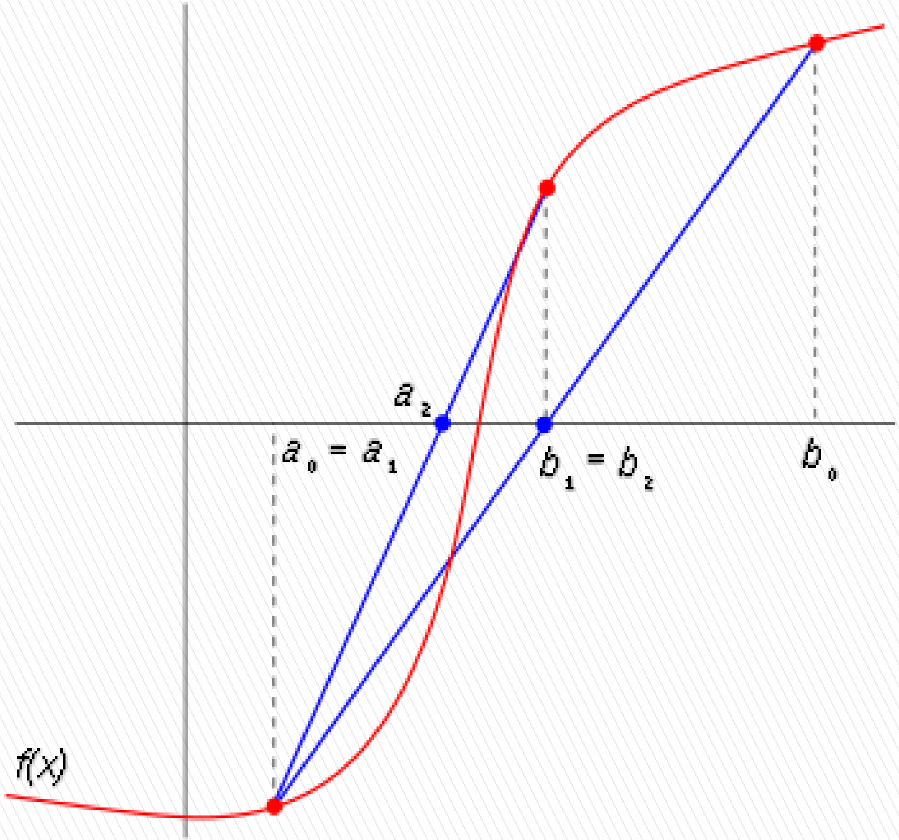
The Method of False Position

- The method of false position is:
  - Similar to the secant method and bisection method
  - Instead of halving the interval [a,b] on which there is a root, we use the root of the secant line through the points (a,f(a)) and (b,f(b)).
- Algorithm: Given $\epsilon > 0$ (tolerance):

1. Choose a and b so that $f(a)f(b)<0$.
2. Draw the secant line that connects (a,f(a)) and (b,f(b)).
3. The point where the secant line crosses the x-axis is c

$$c = b - \frac{f(b)(b-a)}{f(b) - f(a)}$$

The Method of False Position



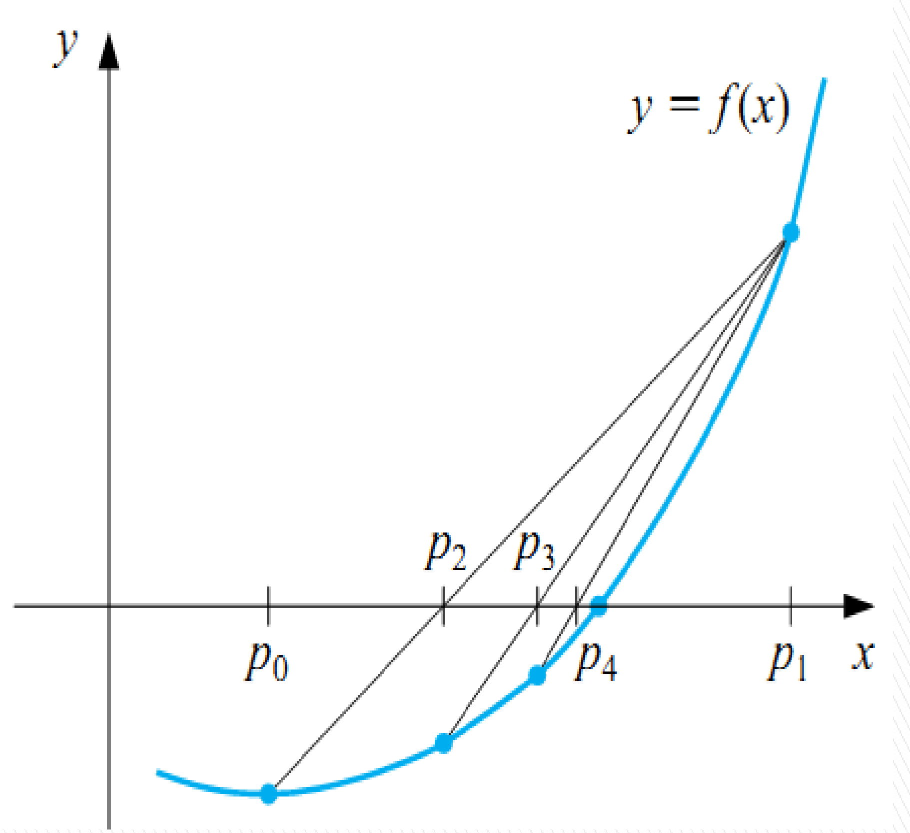
Geometric interpretation of the method of false position.

- If f(c) =0, then we are done.
- If f(a)f(c)<0 then the root must lie in [a,c] so the new interval $[a_1,b_1]=[a,c]$
- If f(a)f(c)>0, then the root must lie in [c,b] so the new interval is $[a_1,b_1]=[c,b]$
- We continue iterating until
  - f(c)=0 or
  - $b_n - a_n < \epsilon$.

The Method of False Position

- The method of false position is easiest to work with when
  - f(x) is concave up (f''(x)>0)
  - f(x) is concave down (f''(x)<0)
- One of the points stays fixed, called false point.
- If p_0 is the false point, the method is given by the formula:

$$p_n = p_{n-1} - \frac{(p_{n-1} - p_0) f(p_{n-1})}{f(p_{n-1}) - f(p_0)}$$



False point – p_1.

Method of False Position: Example

- **Example:** We consider our classical example:
- Find a root of the equation:

$$(x-2)^2 - \ln x = 0$$

on the interval [1.2] accurate within 10^{-5}

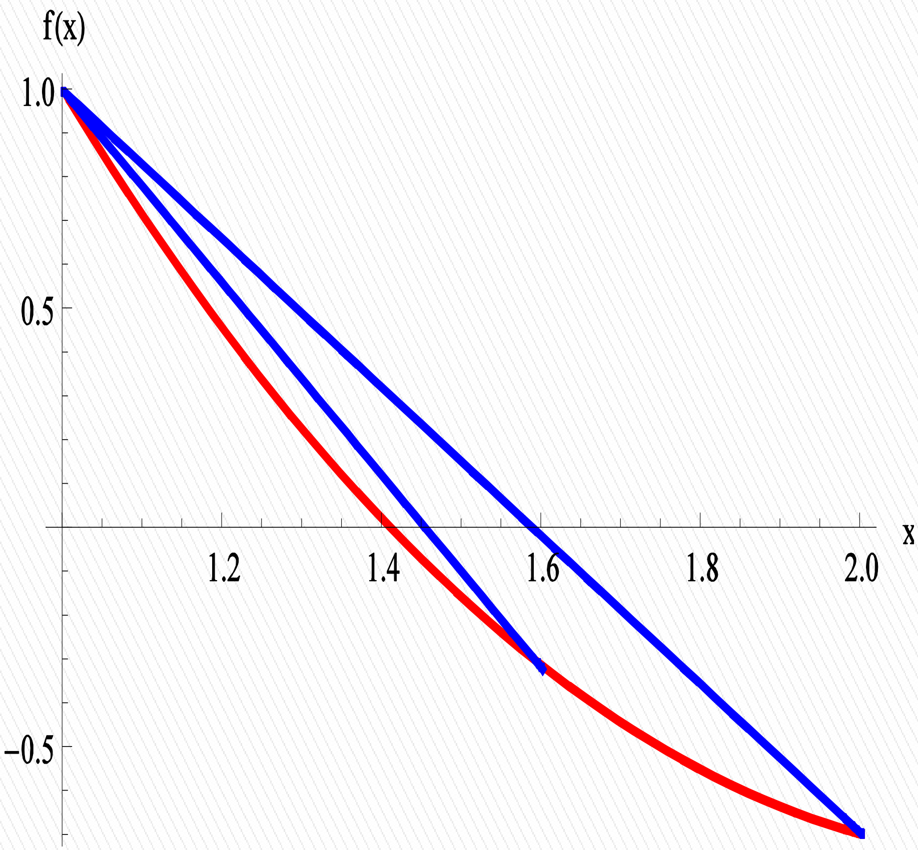
- **Solution**: We use the method of false position. Can the formula be applied?

$$f(x) = (x-2)^2 - \ln x$$

$$f'(x) = 2(x-2) - \frac{1}{x} < 0$$

$$f''(x) = 2 + \frac{1}{x^2} > 0$$

Method of False Position: Example



- f(x) is decreasing and concave up.
- We can use the formula.
- Which point is the false point?
- $p_0=1$ is the false point.
- The formula applies:

$$p_n = p_{n-1} - \frac{(p_{n-1} - p_0) f(p_{n-1})}{f(p_{n-1}) - f(p_0)}$$

Method of False Position: Example

- We run the iteration.
- Accuracy 10^{-5} is reached at iteration p_8.
- The value obtained at iteration 14 is the exact same value obtained by Newton's method at iteration 4.
- Newton's method is much faster.

n	p_n	n	p_n
0	1	8	1.412393726
1	1.5	9	1.412391743
2	1.432726178	10	1.412391299
3	1.416973158	11	1.412391200
4	1,413416651	12	1.412391178
5	1.412620333	13	1.412391173
6	1.412442365	14	1.412391172
7	1.412402607		