2.3. Newton’s Method

1. Newton’s Method
2. The Secant Method
3. The Method of False Position
Problem: Given \( f(x) = 0 \). Find \( x \) in \([a, b]\).

Newton’s Method is one of the most powerful and methods for solving root-finding problems.

Newton’s method is extremely fast, much faster than most iterative methods we can design.

Newton’s method is also called Newton-Raphson method.
Derivation of Newton’s Method from Taylor’s Expansion.

- Suppose $f(x)$ is twice continuously differentiable on $[a,b]$.
- Let $f(p)=0$.
- Let $\eta \approx p$, so that $|p - \eta|$ is small.
  - $f'(\eta) \neq 0$.
- Consider the Taylor’s polynomial expansion of $f(x)$ around $\eta$:

$$f(x) = f(\eta) + f'(\eta)(x-\eta) + f''(\xi(x))(x-\eta)^2/2!$$

Where $\xi(x)$ is a point between $x$ and $\eta$. 
Derivation of Newton’s Method from Taylor’s Expansion

- Set $x = p$ and note that $f(p) = 0$.

$$0 = f(\eta) + f'(\eta)(p - \eta) + f''(\xi(x))(p - \eta)^2/2! \ (\text{ignore!})$$

$$0 \approx f(\eta) + f'(\eta)(p - \eta)$$

Solving for $p$:

$$p \approx \eta - \frac{f(\eta)}{f'(\eta)}$$

- Newton’s method: Given $p_0$ – initial guess of the root, the remaining approximations are computed from

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$$
Newton’s Method

- Newton’s method is a fixed point iteration method:
  \[ p_n = g(p_{n-1}) \]
  where
  \[ g(x) = x - \frac{f(x)}{f'(x)} \]

- Newton’s method cannot continue if for some \( p_{n-1} \)
  \[ f'(p_{n-1}) = 0 \]
Geometrical Interpretation of Newton’s Method

- Choose $p_0$
- Draw the tangent at $(p_0, f(p_0))$
- This tangent crosses the x-axis at $p_1$
- Continue
Use Newton’s method to find the solution accurate to within $10^{-5}$ for the problem

$$(x-2)^2 - \ln x = 0 \quad \text{for } 1 \leq x \leq 2$$

### Solution:

$f(x) = (x-2)^2 - \ln x$

$f'(x) = 2(x-2) - \frac{1}{x}$

The Newton’s method becomes:

$$p_{n+1} = p_n - \frac{(p_n - 2)^2 - \ln p_n}{2(p_n - 2) - \frac{1}{p_n}}$$

Choose $p_0 = 1$ and run the iteration.
# Examples

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Comparing Newton’s Method to Fixed-Point Iterations

- Rewrite the problem

\[(x-2)^2 - \ln x = 0\]

As a number of fixed point iterations. Compare their convergence with Newton’s method.

**Case A:**

\[x = e^{(x-2)^2}\]

\[g(x) = e^{(x-2)^2}\]
Theorem 2.3 does not hold. Conditions b) and d) fail.

The iteration does not converge. Even after 100 iterations, it keeps on jumping from value to value.

### Comparing Newton’s Method to Fixed–Point Iterations

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Comparing Newton’s Method to Fixed-Point Iterations

Case B:

\[(x - 2)^2 = \ln x\]

\[x - 2 = \sqrt{\ln x}\]

\[x = 2 + \sqrt{\ln x}\]

\[g(x) = 2 + \sqrt{\ln x}\]
Comparing Newton’s Method to Fixed-Point Iterations

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This is converging but to a root $p \approx 3.057$ which is not in the interval $[1,2]$.

Oops. What went wrong?

We should have taken the negative root

\[ x - 2 = -\sqrt{\ln x} \]

\[ x = 2 - \sqrt{\ln x} \]

This will converge to a root in the interval $[1,2]$. 
Comparing Newton’s Method to Fixed–Point Iterations

Case C:

\[ x^2 - 4x + 4 = \ln x \]

\[ x = \frac{x^2 - \ln x + 4}{4} \]

\[ g(x) = \frac{x^2 - \ln x + 4}{4} \]

Conditions of Thm 2.3:
1. \( g(x) \) is continuous;
2. \( 1 < g(1) < g(x) < g(2) < 2 \)
3. \( g'(x) \) exists
4. \( g'(x) \) is increasing and positive

\[ |g'(x)| \leq g'(2) = \frac{7}{8} = k < 1 \]
Comparing Newton’s Method to Fixed–Point Iterations

- Thm 2.3 applies so a fixed point iteration will converge for every $p_0$. The rate of convergence is

$$O\left(\frac{7^n}{8}\right)$$

- The relative error

$$\frac{|p_{16} - p_{15}|}{p_{16}} = 2.33 \times 10^{-6}$$

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Theorem 2.5. Let $f(x)$ be twice continuously differentiable on $[a,b]$. If $p$ is such that $f(p)=0$ and $f'(p)\neq 0$ then there exists a $\delta>0$ such that Newton’s method generates a sequence $p_0,p_1,\ldots,p_n,\ldots$ converging to $p$ for any $p_0$ in the interval $[p-\delta,p+\delta]$.

This Thm says that if we start from $p_0$ which is close enough to the root $p$, then Newton’s method will converge.
Importance of the Choice of $p_0$

- **Example**: (showing the importance of $p_0$ in Newton’s method).
- Use Newton’s method to find the solution of $x^3 - 6x^2 + 11x - 12 = 0$ in the interval $[2,5]$.
- **Solution**: The root is $p = 4$. The Newton’s method is given by:

\[
p_{n+1} = p_n - \frac{p_n^3 - 6p_n^2 + 11p_n - 12}{3p_n^2 - 12p_n + 11}
\]
## Importance of the Choice of $p_0$

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Converges in 23 iterations

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Converges in 7 iterations
How Do We Locate $p_0$ close to $p$?

- Suppose the interval $[a,b]$ is large.
- Suppose we don’t know where the root is.
- How do we choose $p_0$ close enough to $p$?

- Run several steps of the bisection method to determine a smaller interval $[a_1,b_1]$ that contains the root.

- In the previous example 2 steps of the bisection method would have given the interval $[3.5,4.25]$. If we chose $p_0$ to be the midpoint, we would have been very close to the root.
Secant Method
Idea of the Secant Method

- Weakness of Newton’s method is that it needs $f'(x)$ which may be difficult to find.
- Secant method computes an approximation of the solution of $f(x)=0$ without the need of $f'(x)$.
- The idea of the secant method is to substitute the slope of the tangent line, given by $f'(p_n)$ with the slope of the secant line through the points $p_{n-1}$ and $p_{n-2}$. 
Recall Newton’s method:

\[ p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} \]

The idea is to replace \( f'(x) \) with the slope of a secant line through \( p_{n-1} \) and \( p_{n-2} \):

\[ f'(p_{n-1}) \approx \frac{f(p_{n-1}) - f(p_{n-2})}{p_{n-1} - p_{n-2}} \]

Replacing the derivative, we obtain the following formula for the Secant method:

\[ p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})} \]
Secant method: Geometrical Interpretation

- We need 2 initial values to start the iteration: $p_0$ and $p_1$.
- We draw the secant line through $p_0$ and $p_1$. We find $p_2$ from the intersection of the secant line with the $x$–axis.
- We draw a new secant line through $p_1$ and $p_2$.
- Continue.
Example for the Secant Method

- **Example**: Using secant method find the solution of the following equation in [1,2].

\[(x - 2)^2 - \ln x = 0\]

- Let \(p_0=1\) and \(p_1=1.5\)

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<th>(p_n)</th>
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Secant method is a little slower than Newton’s method but faster than the bisection method and most fixed-point iterations.

Newton’s method arrived at the value 1.412391172 in 4 iterations.
Method of False Position
The Method of False Position

The method of false position is:

- Similar to the secant method and bisection method
- Instead of halving the interval \([a, b]\) on which there is a root, we use the root of the secant line through the points \((a, f(a))\) and \((b, f(b))\).

Algorithm: Given \(\epsilon > 0\) (tolerance):

1. Choose \(a\) and \(b\) so that \(f(a)f(b) < 0\).
2. Draw the secant line that connects \((a, f(a))\) and \((b, f(b))\).
3. The point where the secant line crosses the \(x\)-axis is \(c\)

\[
c = b - \frac{f(b)(b-a)}{f(b)-f(a)}
\]
The Method of False Position

- If \( f(c) = 0 \), then we are done.
- If \( f(a)f(c) < 0 \) then the root must lie in \([a,c]\) so the new interval is \([a_1, b_1] = [a, c]\)
- If \( f(a)f(c) > 0 \), then the root must lie in \([c, b]\) so the new interval is \([a_1, b_1] = [c, b]\)
- We continue iterating until
  - \( f(c) = 0 \) or
  - \( b_n - a_n < \epsilon \).

Geometric interpretation of the method of false position.
The Method of False Position

- The method of false position is easiest to work with when
  - $f(x)$ is concave up ($f''(x) > 0$)
  - $f(x)$ is concave down ($f''(x) < 0$)
- One of the points stays fixed, called false point.
- If $p_0$ is the false point, the method is given by the formula:

$$p_n = p_{n-1} - \frac{(p_{n-1} - p_0)f(p_{n-1})}{f(p_{n-1}) - f(p_0)}$$
**Example:** We consider our classical example:

Find a root of the equation:

$$(x - 2)^2 - \ln x = 0$$

on the interval $[1,2]$ accurate within $10^{-5}$

**Solution:** We use the method of false position. Can the formula be applied?

$$f(x) = (x - 2)^2 - \ln x$$

$$f'(x) = 2(x - 2) - \frac{1}{x} < 0$$

$$f''(x) = 2 + \frac{1}{x^2} > 0$$
f(x) is decreasing and concave up.

We can use the formula.

Which point is the false point?

p_0 = 1 is the false point.

The formula applies:

\[ p_n = p_{n-1} - \frac{(p_{n-1} - p_0)f(p_{n-1})}{f(p_{n-1}) - f(p_0)} \]
We run the iteration.

Accuracy $10^{-5}$ is reached at iteration $p_8$.

The value obtained at iteration 14 is the exact same value obtained by Newton’s method at iteration 4.

Newton’s method is much faster.