2.4. Error Analysis for Iterative Methods

1. Order of Convergence of a Sequence

- **Definition**: Suppose the sequence

  $p_0, p_1, p_2, \ldots, p_n, \ldots \rightarrow p$

If $\lambda > 0$ and $\alpha > 0$ exist with

$$\lim_{n \to \infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \lambda$$

then the sequence $p_0, p_1, \ldots, p_n, \ldots$ converges to p of order α.

Example: $\frac{1}{100}, \frac{1}{1000}, \frac{1}{10000}, \ldots$ converges with $\alpha = 1$.

$\frac{1}{100}, \frac{1}{10000}, \frac{1}{1000000}, \ldots$ converges with $\alpha = 2$.

Order of Convergence

- **<u>Definition</u>**: λ is called asymptotic error constant.
- Important cases:
  - If $\alpha = 1$ and $0<\lambda\leq 1$, then the sequence is linearly convergent.
  - If $\alpha = 2$, then the sequence is quadratically convergent.
- Example: 1. Show that the sequence $\frac{1}{n^2}$ converges to 0 linearly.

$$\frac{\left|\frac{1}{(n+1)^2}-0\right|}{\left|\frac{1}{n^2}-0\right|^{\alpha}}=\frac{n^{2\alpha}}{(n+1)^2}=\frac{n^{2\alpha}}{n^2+2n+1}-->1$$

Order of Convergence

- 2. Show that the sequence $p_n = 10^{-2^n}$ converges to zero quadratically.

$$\frac{|\, p_{n+1} - 0\,|}{|\, p_n - 0\,|^{\alpha}} = \frac{10^{-2^{n+1}}}{10^{-\alpha * 2^n}} = \frac{10^{-2*2^n}}{10^{-\alpha * 2^n}} -> 1$$

- **<u>Note</u>**: Quadratically convergent sequnce converges much faster

2. Convergence of Fixed Point Iteration

- Consider a sequence generated by a fixed point iteration: $p_{n+1}=g(p_n)$
- <u>**Theorem 2.7**</u>: Let
  - $g(x)$ be continuous of [a,b];
  - $a \le g(x) \le b$ for all x in [a,b];
  - $g'(x)$ exists on (a,b);
  - there exists a constant $0<k<1$ with $|g'(x)| \le k < 1$ for x in (a,b).

  If $g'(p) \neq 0$, then for every p_0 in [a,b] the sequence p_n converges linearly to p.
- We know that the rate of convergence is $O(k^n)$

Convergence of Fixed Point Iteration

- **<u>Theorem 2.8</u>**: Let the first four assumptions of Theorem 2.7 hold. Assume also:
  - $g'(p)=0$
  - $g''(x)$ is continuous on an interval containing p.

  Then, there exists $\delta>0$ such that for p_0 in $[p-\delta,p+\delta]$ the sequence $p_0,p_1,\ldots,p_n,\ldots$ generated by the fixed point iteration

$$p_{n+1}=g(p_n)$$

  converges at least quadratically.

Convergence of Newton's Method

- Consider Newton's method:

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}$$

- Then, $$g(x) = x - \frac{f(x)}{f'(x)}$$

- We compute g'(p), keeping in mind that f(p)=0.

$$g'(x) = 1 - \frac{(f'(x))^2 - f(x)f''(x)}{(f'(x))^2} = \frac{f(x)f''(x)}{(f'(x))^2}$$

Convergence of Newton's Method

- Thus, if $f'(p) \neq 0$, then $g'(p)=0$. Theorem 2.8 implies that Newton's method converges quadratically.
- If $f'(p)=0$ the convergence may not be quadratic.
- It can be shown that the secant method converges of order at least $\alpha=1.618$. So we see it is slower than Newton's method.