Chapter 3: Interpolation and Polynomial Approximation

Introductory Remarks for Chapter 3

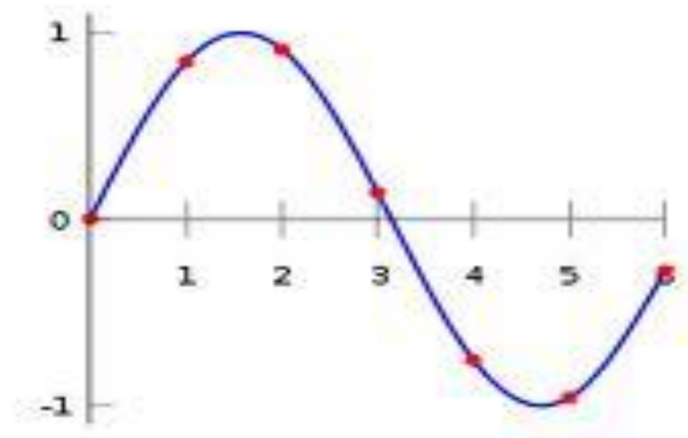
- This chapter is about working with data.
- **Example:** A census of the population of the Us is taken every 10 years.
- If we want to know the population of the US in year 1965 or year 2010, we have to fit a function through the given data.

Year	Population (in thousands)
1940	132,165
1950	151,326
1960	179,323
1970	203,302
1980	226,542
1990	249,633

Goal: To fit functions through data.

Introductory Remarks for Chapter 3

- **<u>Definition</u>**: The process of fitting a function through given data is called interpolation.



- Usually when we have data, we don't know the function f(x) that generated the data. So we fit a certain class of functions.
- The most usual class of functions fitted through data are polynomials. We will see why polynomials are fitted through data when we don't know f(x).

Introductory Remarks for Chapter 3

- **Definition**: The process of fitting a polynomial through given data is called polynomial interpolation.
- Polynomials are often used because they have the property of approximating any continuous function.
- Given:
  - f(x) continuous on [a,b]
  - $\epsilon > 0$ (called tolerance)

  Then, there is a polynomial P(x) of appropriate degree which approximates the function within the given tolerance.

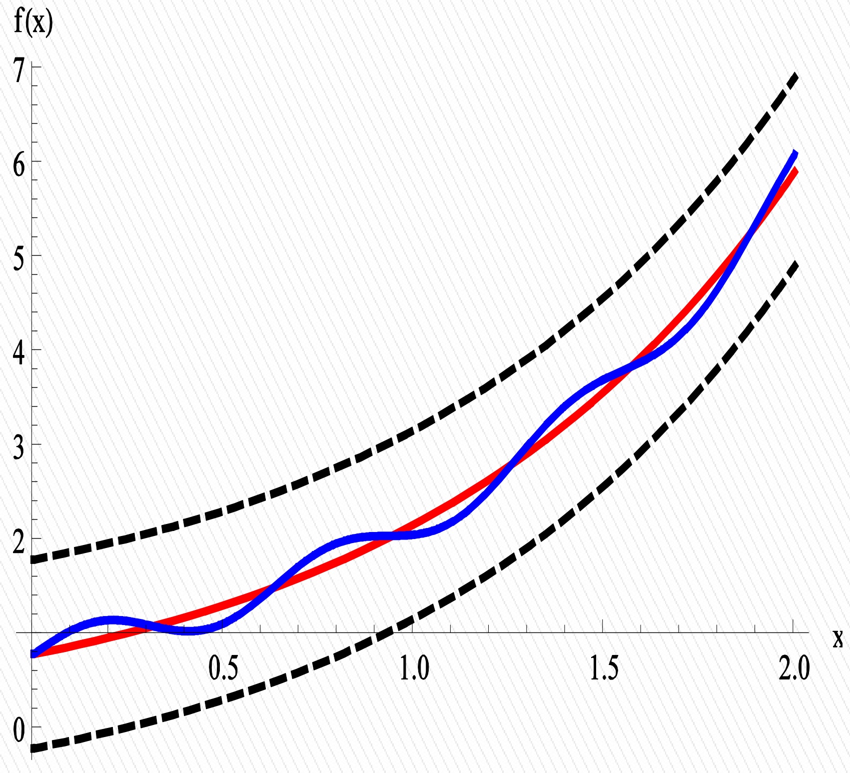
Introductory Remarks for Chapter 3

- This fact is guaranteed by the following Theorem:

Theorem: (Weierstrass Approximation Theorem): Suppose that f(x) is defined and continuous on [a,b]. For each $\epsilon>0$, there exists a polynomial P(x) with the property

$$|f(x)-P(x)|<\epsilon$$

for all x in [a,b].



f(x) in red approximated by P(x) in blue within ϵ.

3.1 Interpolation and the Lagrange Polynomial

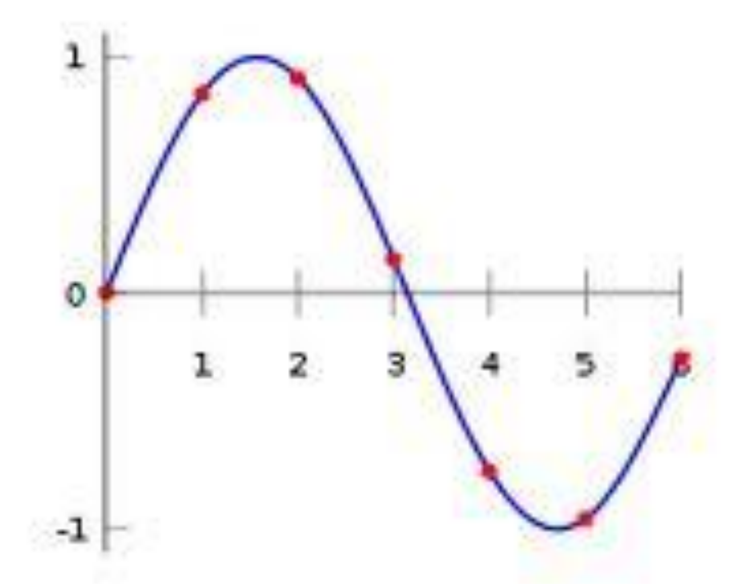
1. Lagrange Interpolating Polynomial

- **Problem**: Given the values of the function f(x) at n+1 distinct points
- x_0 x_1 x_2, …, x_n
- f_0 f_1 f_2,…., f_n

  where $f_i = f(x_i)$ for $i=0,\ldots,n$.

  Find a polynomial of degree n, P(x), such that

  $P(x_i)=f_i$, $i=0,\ldots,n$



Polynomial of degree n has n+1 coefficients, that is n+1 unknowns to determine. We have n+1 conditions given.

$$P(x) = a_n x^n + \ldots + a_1 x + a_0$$

Simplest Case: 2 points

- When n=2, the problem becomes: Given:
- x_0 x_1
- f_0 f_1

Find a polynomial of degree one such that

$P(x_0)=f_0$ $P(x_1)=f_1$

$$y = f_0 + \frac{f_1 - f_0}{x_1 - x_0}(x - x_0) = f_0 + (f_1 - f_0)\frac{x - x_0}{x_1 - x_0} =$$

$$= f_0 \frac{x - x_1}{x_0 - x_1} + f_1 \frac{x - x_0}{x_1 - x_0}$$

Simplest Case: 2 points

- Denote by $L_0(x)$ and $L_1(x)$ the two first degree polynomials:

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} \qquad L_1(x) = \frac{x - x_0}{x_1 - x_0}$$

- These polynomials have the following properties:
- $L_0(x_0)=1 \qquad L_0(x_1)=0$
- $L_1(x_0)=0 \qquad L_1(x_1)=1$
- So we can rewrite the polynomial that fits the data in the form:

$$P(x) = f_0 L_0(x) + f_1 L_1(x)$$

Special Case: 2 points

- $P(x_0)=f_0L_0(x_0)+f_1L_1(x_1)= \ f_0$
- $P(x_1)=f_0L_0(x_1)+f_1L_1(x_1) = f_1$

- $L_0(x_0)=1, \ L_0(x_1)=0$
- $L_1(x_0)=0, \ L_1(x_1)=1$

- Note: $P(x)$ is the <u>unique</u> linear polynomial passing through the points (x_0,f_0) and (x_1,f_1)

The General Case: n+1 points

- Now we generalize the approach to n+1 points.
- Given:
  - $x_0, x_1, x_2, \ldots, x_n$
  - f_0, f_1, $f_2, \ldots, f_n$
- First we construct the special polynomials $L_k(x)$ so that they have the properties:
  - $L_k(x_i)=0$ if $i \neq k$
  - $L_k(x_k)=1$
- These are polynomials that are zero at all points except one.

The General Case: n+1 points

- A polynomial of degree n which is zero at all points except x_k is given by
  - $(x-x_0)(x-x_1)\ldots(x-x_{k-1})(x-x_{k+1})\ldots(x-x_n)$
- If we want that polynomial to have value one at x_k we must divide by its value at x_k. Thus,

$$L_k(x) = \frac{(x-x_0)\ldots(x-x_{k-1})(x-x_{k+1})\ldots(x-x_n)}{(x_k-x_0)\ldots(x_k-x_{k-1})(x_k-x_{k+1})\ldots(x_k-x_n)}$$

- $L_k(x)$ is called basic Lagrange polynomial of degree n.

The General Case: n+1 points

- Then $P(x) = f_0L_0(x) + f_1L_1(x) + \dots f_nL_n(x)$
- We would have $P(x_k) = f_k$ for k=0,…,n.
- **Definition**: P(x) is called the nth Lagrange interpolating polynomial.

- **Example**: Find the appropriate Lagrange interpolating polynomial using the table:

x	0	0.5	1	1.5
f(x)	1	2	3	4

2. Example

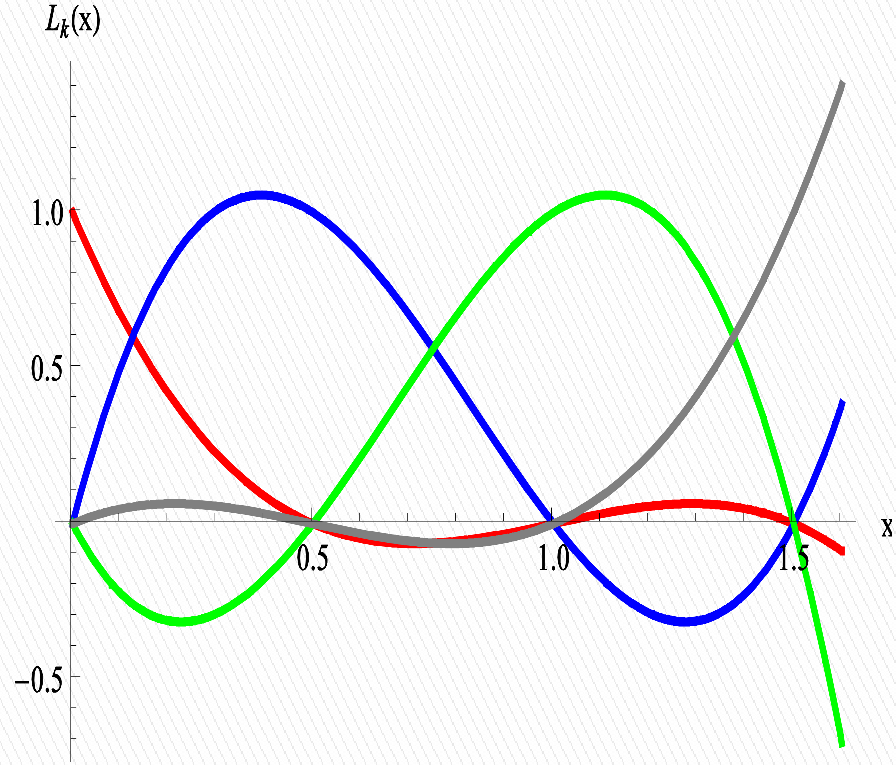
- We have 4 points, so the Lagrange polynomial will be of degree 3.
- The basic Lagrange polynomials are:

$$L_0(x) = \frac{(x-0.5)(x-1)(x-1.5)}{(0-0.5)(0-1)(0-1.5)}$$

$$L_1(x) = \frac{x(x-1)(x-1.5)}{0.5(0.5-1)(0.5-1.5)}$$

$$L_2(x) = \frac{x(x-0.5)(x-1.5)}{1(1-0.5)(1-1.5)}$$

$$L_3(x) = \frac{x(x-0.5)(x-1)}{1.5(1.5-0.5)(1.5-1)}$$



$P(x) = L_0(x) + 2L_1(x) + 3L_2(x) + 4L_3(x)$

3. The Interpolation Error

- If we replace the function $f(x)$ with the polynomial $P(x)$, we would like to know what error we are making.
- **Definition**: The error $E_n(x,f) = f(x) - P(x)$
- The error varies from point to point. In the interpolation points the error is zero but it is non-zero in other points.
- We will be most interested in the maximum of the $|E_n(x,f)|$ over the interval $[a,b]$. This maximum is called error bound.

The Interpolation Error

- **Theorem 3.3**: Suppose $x_0, x_1, \ldots, x_n$ are n+1 distinct points in the interval [a,b] and f(x) has n+1 continuous derivatives. Then, for each x in [a,b], a number ξ(x) in (a,b) exists with

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)\ldots(x - x_n)$$

where P(x) is the nth Lagrange interpolating polynomial.

The Interpolation Error

- **Definition**: The error is given by the formula:

$$E_n(x, f) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)\ldots(x - x_n)$$

- Note:
  - If the interval [a,b] is not given, we take $[a,b]=[x_0,x_n]$.
  - We need the function f(x) in order to be able to compute an error bound.

Example 1

- **Example**: For the function

$$f(x)=\cos(x)$$

let $x_0=0$, $x_1=0.6$, $x_2=0.9$.

  - Construct the interpolation polynomial of degree at most two to approximate $f(0.45)$.
  - Find the actual error at 0.45.
  - Use Theorem 3.3 to find the error bound for the error.

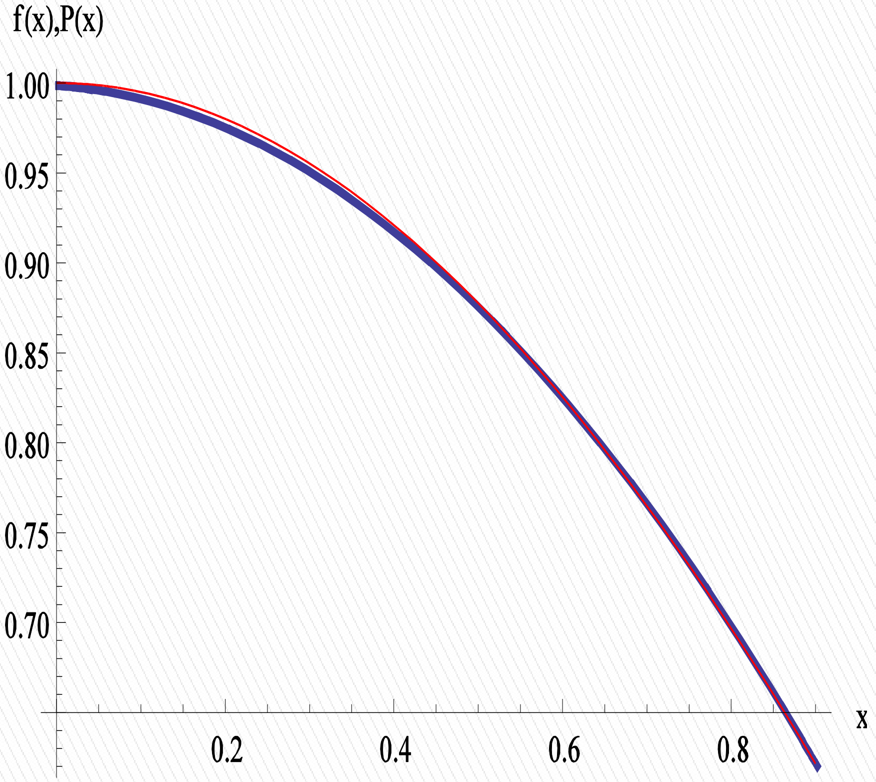
Example 1 – Solution

- The basic Lagrange polynomials are:

$$L_0(x) = \frac{(x-0.6)(x-0.9)}{(0-0.6)(0-0.9)}$$

$$L_1(x) = \frac{x(x-0.9)}{0.6(0.6-0.9)}$$

$$L_2(x) = \frac{x(x-0.6)}{0.9(0.9-0.6)}$$



$P(x) = L_0(x) + \cos(0.6) L_1(x) + \cos(0.9) L_2(x)$

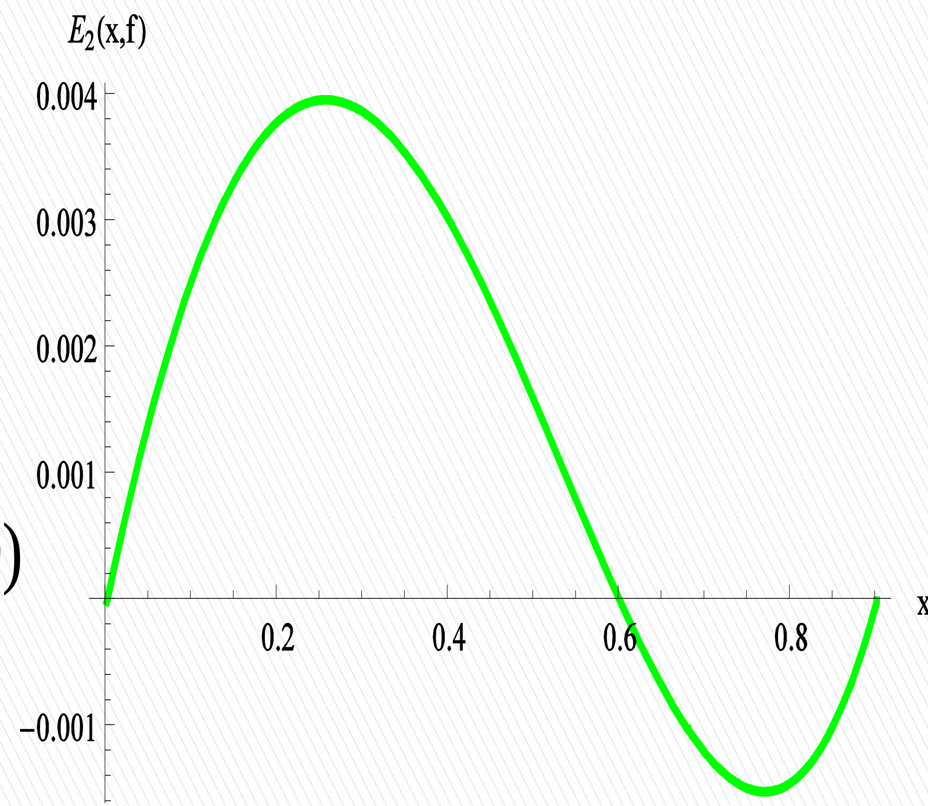
Plot of f(x) and P(x).

Example 1 – Solution

- P(0.45)=0.898100747
- cos(0.45) = 0.9004471024
- Actual error at x=0.45:
  cos(0.45)−P(0.45)≈0.0023
- Error bound:

$$E_2(x,f)=\frac{f'''(\xi(x))}{3!}x(x-0.6)(x-0.9)$$

- We want to compute
  $\max_x |E_2(x,f)|$



Plot of the error.

Example 1 – Solution

- We can see from the figure that

$$\max_x |E_2(x,f)| \leq 0.4$$

- However, we will not be able to compute this best bound because we don't know the location of ξ.
- We will bound the error in 2 steps:
  - Find

$$\max_{a \leq x \leq b} | f'''(x) |$$

  - Find

$$\max_{a \leq x \leq b} | x(x-0.6)(x-0.9) |$$

Example 1 – Solution

- Computing

$$\max_{a \le x \le b} | f'''(x) |$$

- Computing derivatives:

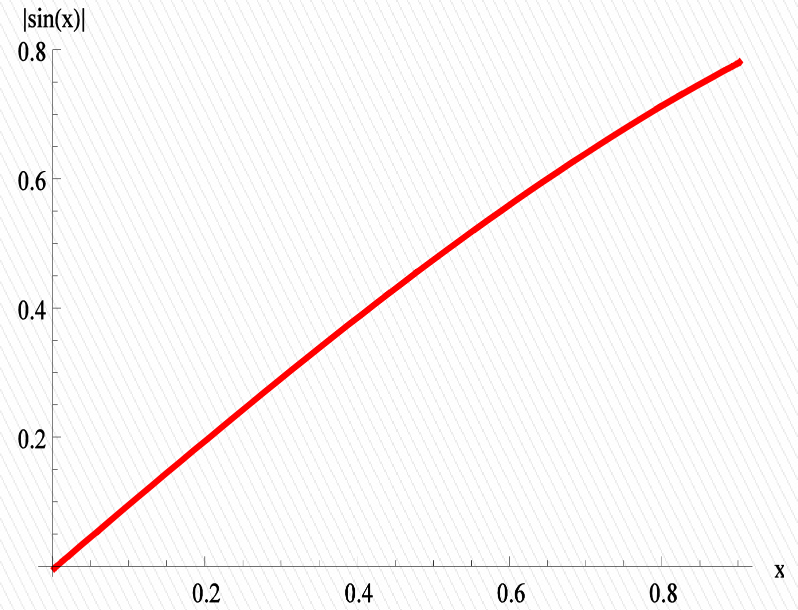
  f(x) = cos x

  f'(x) = -sin x

  f''(x) =-cos x

  f'''(x)=sin x

- |sin(x)| is increasing over [0,0.9]. Hence:



$$\max_{a \le x \le b} | \sin(x) | \le | \sin(0.9) |$$

Example 1 – Solution

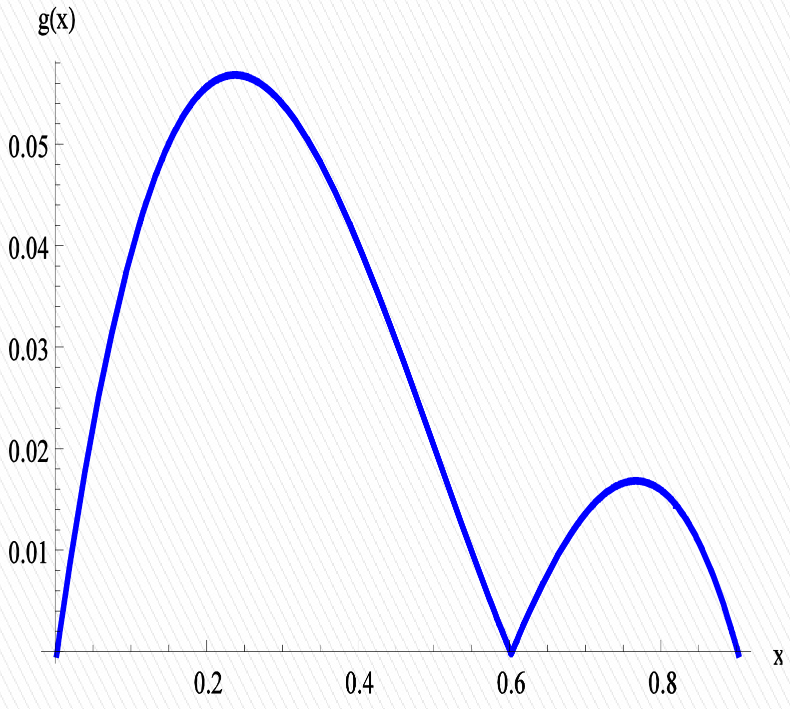
- Computing

$$\max_{a\le x\le b} |x(x-0.6)(x-0.9)|$$

- g(x)=x(x−0.6)(x−0.9)

$$g'(x)=3(x^2-x+0.18)=0$$

- $p_1=0.2354248689$
- $p_2=0.7645751311$



$|g(x)| \le |g(p_1)| = 0.05704$

$|g(p_2)| = 0.0170405184$

Example 1 – Solution

- We put together the two estimates:

$$|E_2(x,f)| = \frac{|\sin(\xi(x))|}{6} |x(x-0.6)(x-0.9)|$$

$$\le \frac{|\sin 0.9|}{6} 0.05704 = 0.0074468$$

Example 2

- Construct the Lagrange interpolating polynomial of degree 2 for

$$f(x)=\sin(\ln x)$$

on the interval [2,2.6] with the points

$$x_0=2.0 \quad x_1=2.4 \quad x_2=2.6.$$

Find a bound for the absolute error.

- Thus, the following table is given:

x	2	2.4	2.6
f(x)	sin(ln2)	sin(ln2.4)	sin(ln2.6)

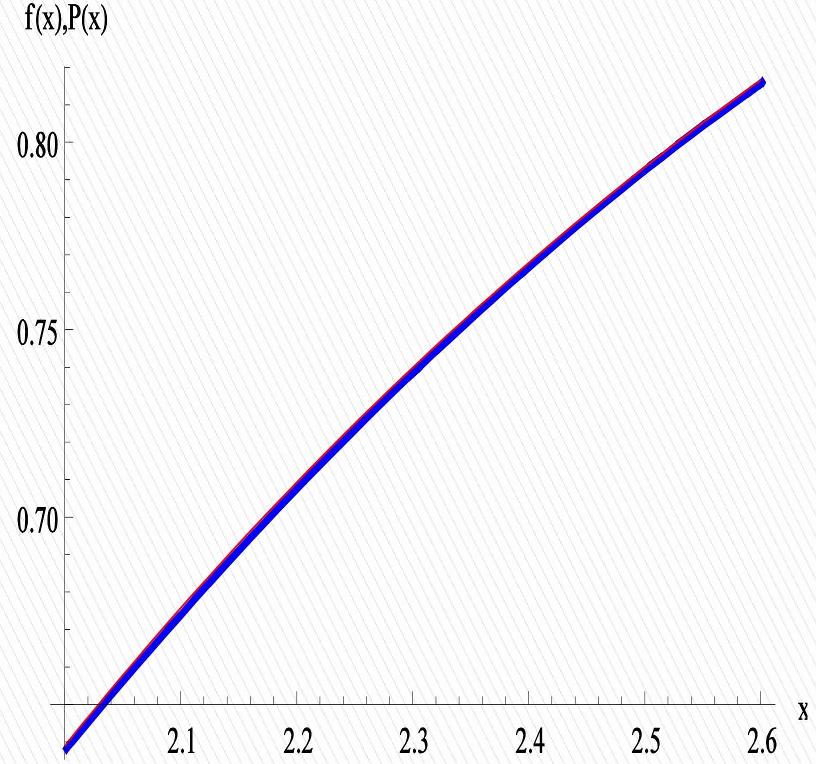
Example 2 – Solution

- We construct the basic Lagrange polynomials:

$$L_0(x) = \frac{(x-2.4)(x-2.6)}{(2-2.4)(2-2.6)}$$

$$L_1(x) = \frac{(x-2)(x-2.6)}{(2.4-2)(2.4-2.6)}$$

$$L_2(x) = \frac{(x-2)(x-2.4)}{(2.6-2)(2.6-2.4)}$$



$P(x)=\sin(\ln 2)L_0(x)+\sin(\ln 2.4)L_1(x)+\sin(\ln 2.6)L_2(x)$

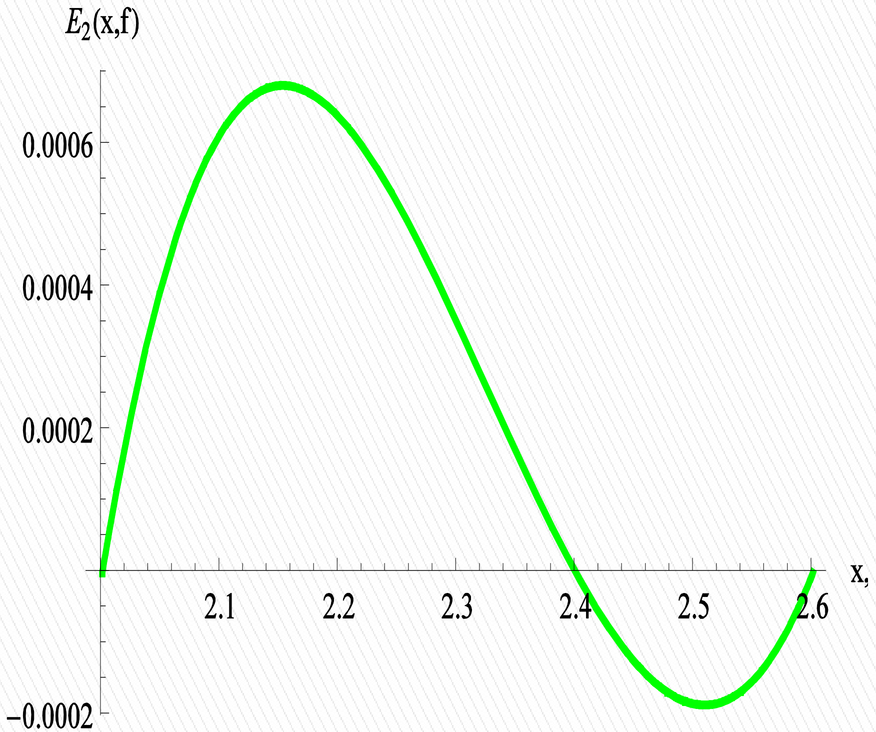
Example 2 – Solution

- The error is given by

$$E_2(x,f)=\frac{f'''(\xi(x))}{3!}(x-2)(x-2.4)(x-2.6)$$

f(x)=sin(lnx))

$$f'''(x)=\frac{3\sin(\ln x)+\cos(\ln x)}{x^3}$$



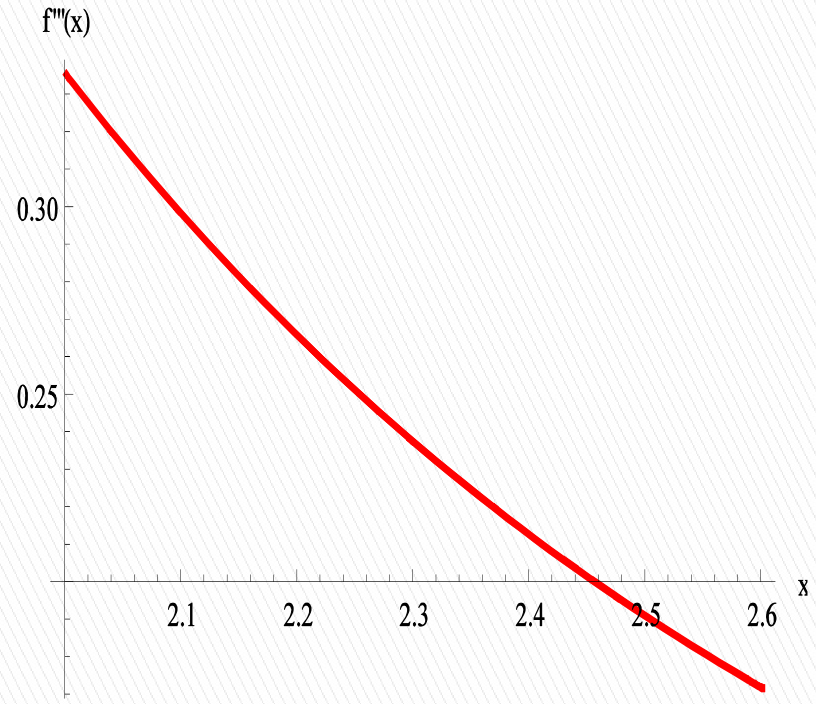
Plot of the error f(x)–P(x)

Example 2 – Solution

- Bounding the third derivative:
- |f’’’(x)| is plotted on the right. |f’’’(x)| is decreasing. The maximum is at 2:

$$\max_{2\le x\le 2.6} |f'''(x)| \le |f'''(2)| = $$

$$= \frac{3\sin(\ln 2)+\cos(\ln 2)}{8} = 0.335765$$



f’’’(x) plotted over the interval [2,2.6].

Example 2 – Solution

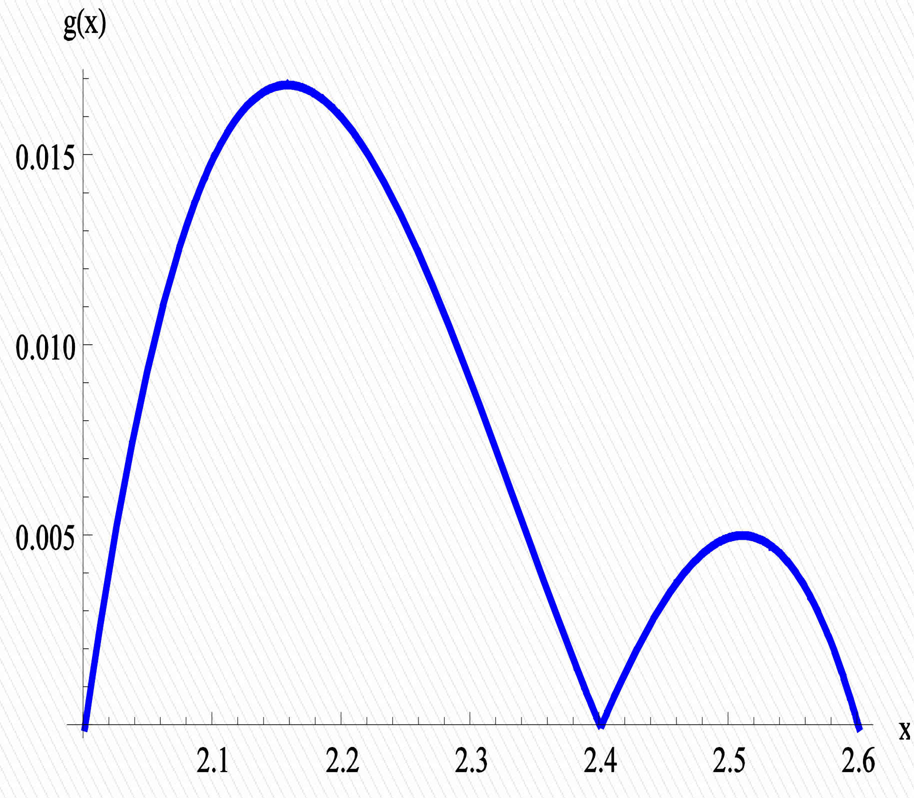
- Next we bound $|g(x)|$:

$g(x)=(x-2)(x-2.4)(x-2.6)$

- To find the maximum of $|g(x)|$, we need to take the derivative:

$$g'(x)=3x^2-14x+16.24=0$$

- $p_1=2.157$
- $p_2=2.5$



$|g(x)|\leq|g(2.157)|=0.0169$
$|g(2.5)|=0.005$

Plot of g(x).

Example 2 – Solution

- Obtaining the error bound:

$$|E_2(x,f)| = \frac{|f'''(\xi(x))|}{3!}|(x-2)(x-2.4)(x-2.6)|$$

$$\leq \frac{0.335765}{6}0.0169 = 9.457*10^{-4}$$