3.2. Divided Differences

Introduction

- Why should Lagrange polynomial interpolation method be improved?
  - A practical difficulty with Lagrange interpolation is that since the error term is difficult to apply, the degree of the interpolating polynomial is NOT known until after the computation.
  - The work done in calculating the nth degree polynomial does not lessen the work for the computation of the (n+1)st degree polynomial
- To remedy these problems Newton created a different approach to the same problem of interpolating (n+1) points.

Problem:

- We are solving the same problem:
- Given

$x_0 \quad x_1 \qquad\qquad x_n$

$f_0 \quad f_1 \qquad\qquad f_n$

find a polynomial of degree at most n, P(x), that goes through all the points, that is satisfies:

$$P(x_k)=f_k$$

- We take a new approach to this problem.

1. Divided Differences

- Let $P_n(x)$ be the nth degree interpolating polynomial. We want to rewrite Pn(x) in the form

$$P_n(x)=a_0+a_1(x-x_0)+a_2(x-x_0)(x-x_1)+ \\ \ldots.+a_n(x-x_0)(x-x_1)\ldots(x-x_{n-1})$$

for appropriate constants $a_0,a_1,\ldots,a_n$.

- We want to determine the coefficients $a_0,a_1,\ldots,a_n$.
- Determining a_0 is easy: $a_0=P_n(x_0)=f_0$

Divided Differences

- To determine a_1 we compute

$$P_n(x_1)=a_0+a_1(x-x_0)$$

$$f_1=f_0+a_1(x_1-x_0)$$

- Solving for a_1 we have

$$a_1=\frac{f_1-f_0}{x_1-x_0}$$

- This prompts to define the coefficients to be the divided differences.
- The divided differences are defined recursively.

Divided Differences

- **Definition**: The 0th divided difference of a function f with respect to the point x_i is denoted by $f[x_i]$ and it is defined by

$$f[x_i]=f(x_i)$$

- **Definition:** The first divided difference of f with respect to x_i, x_{i+1} is denoted by $f[x_i,x_{i+1}]$ and it is defined as follows:

$$f[x_i,x_{i+1}]=\frac{f[x_{i+1}]-f[x_i]}{x_{i+1}-x_i}$$

Divided Differences

- **Definition**: The second divided difference at the points x_i, x_{i+1}, x_{i+2} denoted by $f[x_i, x_{i+1}, x_{i+2}]$ is defined as follows:

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$$

- **Definition**: If the (k−1)st divided differences

  $f[x_i, \ldots, x_{i+k-1}]$ and $f[x_{i+1}, \ldots, x_{i+k}]$

  are given, the kth divided difference relative to $x_i, \ldots, x_{i+k}$ is given by

$$f[x_i, \ldots, x_{i+k}] = \frac{f[x_{i+1}, \ldots, x_{i+k}] - f[x_i, \ldots, x_{i+k-1}]}{x_{i+k} - x_i}$$

Divided Differences

- The divided differences are computed in table:

x	f(x)	Ist DD	IInd DD	IIIrd DD	IVth DD
x_0	f_0				
x_1	f_1	$f[x_0,x_1]$			
x_2	f_2	$f[x_1,x_2]$	$f[x_0,x_1,x_2]$		
x_3	f_3	$f[x_2,x_3]$	$f[x_1,x_2,x_3]$	$f[x_0,x_1,x_2,x_3]$	
x_4	f_4	$f[x_3,x_4]$	$f[x_2,x_3,x_4]$	$f[x_1,x_2,x_3,x_4]$	$f[x_0,x_1,x_2,x_3,x_4]$
…	….	….	…	….	

Example

- Compute the divided differences with following data:

x	f(x)			
0	3			
1	4			
2	7			
4	19			

Example

- Completing the table:

x	f(x)	Ist DD	IInd DD	IIIrd DD
0	3			
1	4	1		
2	7	3	1	
4	19	6	1	0

2. Interpolating with Divided Differences

- If we want to write the interpolating polynomial in the form

$$P_n(x)=a_0+a_1(x-x_0)+a_2(x-x_0)(x-x_1)+\ldots+a_n(x-x_0)(x-x_1)\ldots(x-x_{n-1})$$

we saw that

$$a_0 = f(x_0) = f_0 = f[x_0]$$

$$a_1 = \frac{f(x_1)-f(x_0)}{x_1-x_0} = f[x_0,x_1]$$

- If we continue to compute we will get:

$$a_k=f[x_0,x_1,\ldots,x_k]$$

for all $k=0,1,\ldots,n$.

Interpolating with Divided Differences

- So the nth interpolating polynomial becomes:

$P_n(x)=f[x_0]+f[x_0,x_1](x-x_0)+f[x_0,x_1,x_2](x-x_0)(x-x_1)+\ldots+f[x_0,\ldots,x_n](x-x_0)\ldots(x-x_{n-1})$

- **Definition**: This formula is called Newton's interpolatory forward divided difference formula.

- **Example**: (A) Construct the interpolating polynomial of degree 4 for the points:

x	0.0	0.1	0.3	0.6	1.0
f(x)	-6.0000	-5.89483	-5.65014	-5.17788	-4.28172

Example

- We construct the divided difference table

x	f(x)	Ist DD	IInd DD	IIIrd DD	IVth DD
0.0	−6.00000				
0.1	−5.89483	1.0517			
0.3	−5.65014	1.22345	0.5725		
0.6	−5.17788	1.5742	0.7015	0.215	
1.0	−4.28172	2.2404	0.9517	0.278	0.063

- Then, Newton's forward polynomial is:

$$P_4(x) = -6 + 1.0517x + 0.5725x(x-0.1) + 0.215x(x-0.1)(x-0.3) + 0.063x(x-0.1)(x-0.3)(x-0.6)$$

Example

- (B) Add the point $f(1.1)=-3.99583$ to the table, and construct the polynomial of degree five.

x	f(x)	Ist DD	IInd DD	IIIrd DD	IVth DD	Vth DD
0.0	–6.00000					
0.1	–5.89483	1.0517				
0.3	–5.65014	1.22345	0.5725			
0.6	–5.17788	1.5742	0.7015	0.215		
1.0	–4.28172	2.2404	0.9517	0.278	0.063	
1.1	–3.99583	2.8589	1.237	0.356625	0.078625	0.0142

- Newton's polynomial: $P_5(x)=P_4(x)+$

$$+0.0142x(x-0.1)(x-0.3)(x-0.6)(x-1)$$

Newton's Backward Formula

- If the interpolating nodes are reordered as

  $x_n, x_{n-1}, \ldots x_1, x_0$

  a formula similar to the Newton's forward divided difference formula can be established.
- $P_n(x) = f[x_n] + f[x_n, x_{n-1}](x - x_n) + \ldots$

  $+ f[x_n, \ldots, x_0](x - x_n) \ldots (x - x_1)$
- **<u>Definition</u>**: This formula is called Newton's backward divided difference formula.

Example

- Construct the interpolating polynomial of degree four using Newton's backward divided difference formula using the data:

0.0	-6.00000				
0.1	-5.89483	1.0517			
0.3	-5.65014	1.22345	0.5725		
0.6	-5.17788	1.5742	0.7015	0.215	
1.0	-4.28172	2.2404	0.9517	0.278	0.063

$$P_4(x) = -4.28172 + 2.2404(x-1) + $$
$$+0.9517(x-1)(x-0.6) +$$
$$+0.278(x-1)(x-0.6)(x-0.3)$$
$$+0.063(x-1)(x-0.6)(x-0.3)(x-0.1)$$

3. Error of Interpolation with Divided Differences

- The nth degree polynomial generated by the Newton's divided difference formula is the **exact same polynomial** generated by Lagrange formula. Thus, the error is the same:

$$E_n(x, f) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)\ldots(x - x_n)$$

- Recall also that

$$E_n(x,f) = f(x) - P_n(x)$$

Example

- For the function

$$f(x) = x^2 e^{\frac{-x}{2}}$$

  - Construct the divided difference table for the points
    $x_0=1.1$ $x_1=2$ $x_2=3.5$ $x_3=5$ $x_4=7.1$
  - Find the Newton's forward divided difference polynomials of degree 1, 2 and 3.
  - Find the errors of the interpolates for f(1.75).
  - Find the error bound for $E_1(x,f)$.

Example – Solution

- The divided difference table is:

1.75

x	f(x)	Ist DD	IInd DD	IIIrd DD	IVth DD
1.1	0.6981				
2	1.4715	0.8593			
3.5	2.1287	0.4381	–0.1755		
5	2.0521	–0.0511	–0.1631	0.0032	
7.1	1.4480	–0.2877	–0.0657	0.0191	0.0027

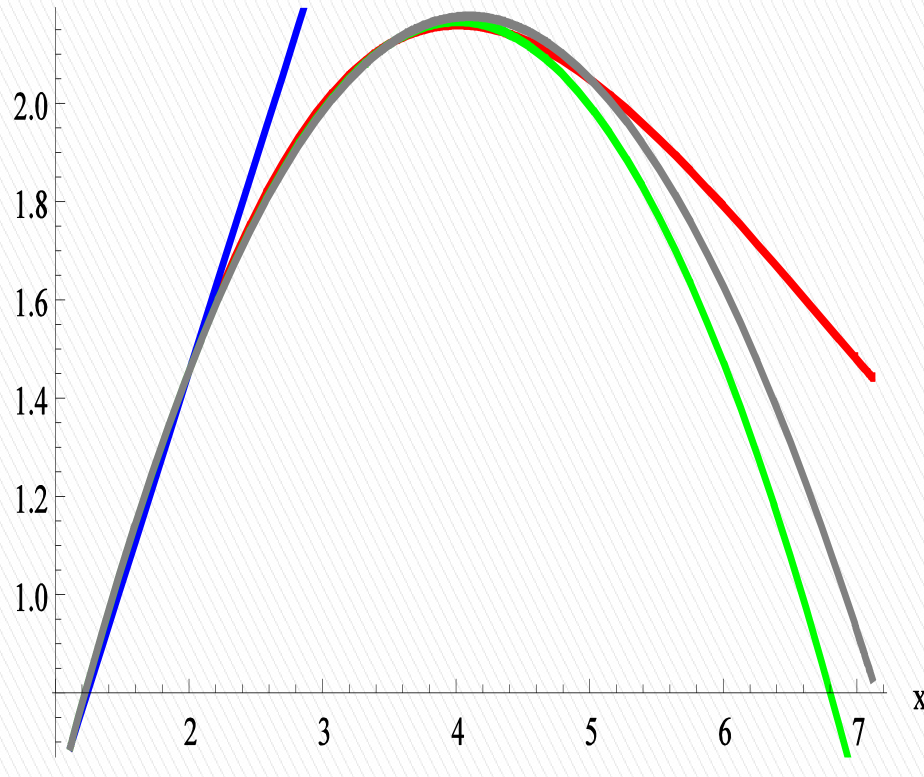
- $P_1(x) = 0.6981 + 0.8593(x-1.1)$
- $P_2(x) = P_1(x) - 0.1755(x-1.1)(x-2)$
- $P_3(x) = P_2(x) + 0.0032(x-1.1)(x-2)(x-3.5)$

Example – Solution

- $f(1.75)=1.2766$

Degree	$P_n(1.75)$	Actual error
1	1.25665	0.01995
2	1.2852	-0.0086
3	1.2861	-0.0095

- Typically we can expect that a higher degree polynomial will approximate better but here $P_2(x)$ approximates better than $P_3(x)$.
- Difference is small.



$f(x)$ in red, $P_1(x)$ in blue, $P_2(x)$ in green, $P_3(x)$ in gray

Example – Bounding the Error

- The error of $P_1(x)$ is

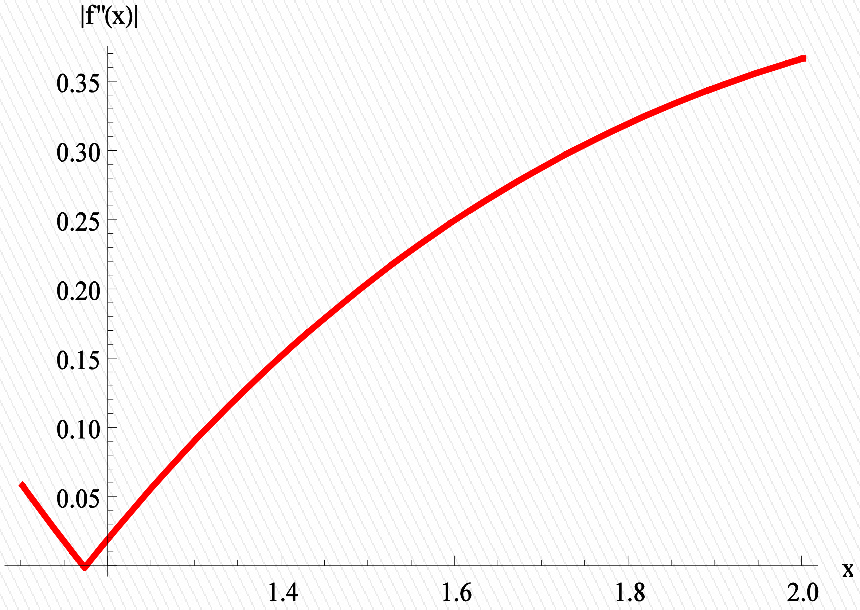
$$E_1(x, f) = \frac{f''(\xi(x))}{2!}(x-1.1)(x-2)$$

- We find the derivatives

$$f(x) = x^2 e^{-\frac{x}{2}}$$

$$f'(x) = (2x - \frac{x^2}{2})e^{-\frac{x}{2}}$$

$$f''(x) = (2 - 2x + \frac{x^2}{4})e^{-\frac{x}{2}}$$



$\max_x |f''(x)| \le |f''(2)| = 0.3679$

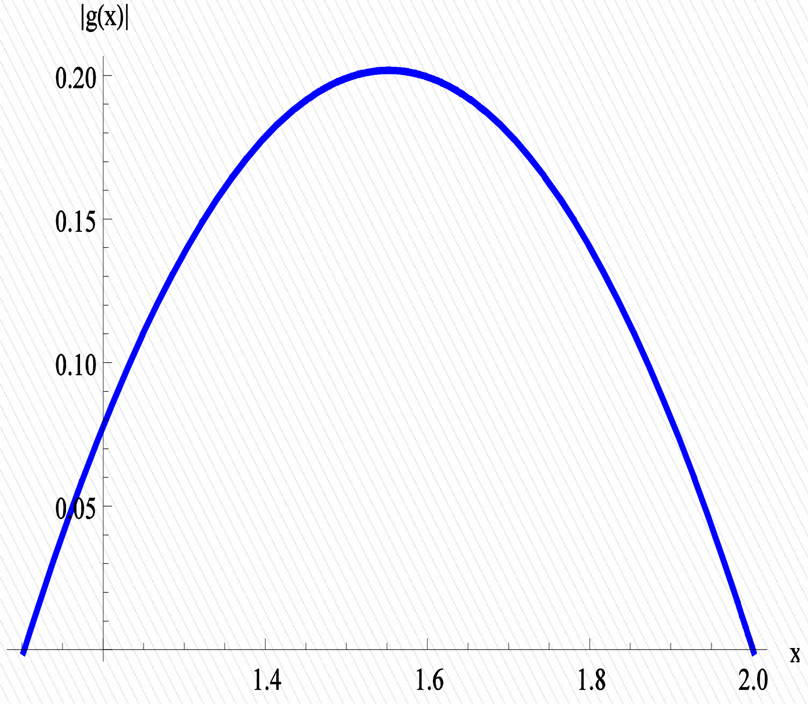
Plot of $|f''(x)|$ on [1.1,2]

Example – Bounding the Error

- $g(x)=(x-1.1)(x-2)$
- The maximum of $|g(x)|$ is attained at the midpoint of the interval [1.1,2]:
- $p_m=(1.1+2)/2=1.55$
- $|g(x)|\le|g(1.55)|=0.2025$
- Error bound:

$$|E_1(x,f)|=\frac{|f''(\xi(x))|}{2!}|(x-1.1)(x-2)|$$

$$\le\frac{0.3679}{2}0.2025=0.03725$$



Plot of |g(x)| on [1.1,2].

How Does the Divided Difference Relate to the Derivative?

- Notice that

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

- The Mean Value Theorem says that if $f'(x)$ exists, then

$$f[x_0, x_1] = f'(\xi)$$

for some ξ between x_0 and x_1.

How Does the Divided Difference Relate to the Derivative?

- The following Theorem generalizes this:

- **Theorem 3.6**: Suppose f has n continuous derivatives and x0,x1,…,xn are distinct numbers in [a,b]. Then ξ in (a,b) exists with

$$f[x_0,\ldots,x_n]=\frac{f^{(n)}(\xi)}{n!}$$

Error Estimation when f(x) is Unknown: Next Term Rule

- Often f(x) is NOT known, and the nth derivative of f(x) is also not known. Therefore, it is hard to bound the error.
- We saw that

$$f[x_0,\ldots,x_n]=\frac{f^{(n)}(\xi)}{n!}$$

- Thus, the nth divided difference is an estimate of the nth derivative of f.

Error Estimation when f(x) is Unknown: Next Term Rule

- This means that the error is approximated by the value of the next term to be added:

$$E_n(x, f) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)\dots(x - x_n)$$

$$\approx f[x_0, \dots, x_n, x_{n+1}](x - x_0)\dots(x - x_n)$$

- $E_n(x,f) \approx$ the value of the next term that would be added to $P_n(x)$.

Example – Next Term Rule

- For the function

$$f(x) = x^2 e^{-\frac{x}{2}}$$

  - Construct the divided difference table for the points
    $x_0=1.1$ $x_1=2$ $x_2=3.5$ $x_3=5$ $x_4=7.1$
  - Find the Newton's forward divided difference polynomial of degree 1.
  - Use the next term rule to estimate the error of the interpolate for f(1.75).

Example – Next Term Rule

- The divided difference table is:

x	f(x)	Ist DD	IInd DD
1.1	0.6981		
2	1.4715	0.8593	
3.5	2.1287	0.4381	−0.1755

- $P_1(x) = 0.6981 + 0.8593(x-1.1)$
- $P_2(x) = P_1(x) - 0.1755(x-1.1)(x-2)$

- The next term rule gives:
- $E_1(1.75, f) \approx -0.17755(1.75-1.1)(1.75-2) = 0.02852$

4. Interp. With Equally Spaced Points. Ordinary Differences

- **Definition**: The points $x_0, x_1, \ldots, x_n$ are called equally spaced if

  $$x_1 - x_0 = x_2 - x_1 = \ldots = x_n - x_{n-1} = h \text{ (step)}.$$

- **Example**: $x_0 = 1 \quad x_1 = 1.5 \quad x_2 = 2 \quad x_3 = 2.5$
- If the data are equally spaced getting the interpolation polynomial is simpler.
- When we compute the divided differences we will always divide by the same number.
- In this case it is more convenient to define ordinary differences.

Ordinary Differences

- **Definition**: The first forward difference $\Delta f(x_i)$ is defined as

$$\Delta f(x_i) = f(x_{i+1}) - f(x_i)$$

- Then,

$$f[x_i, x_{i+1}] = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} = \frac{\Delta f(x_i)}{h}$$

- **Example**: Let f(x)=ln(x). The first forward difference at the points $x_0=1$ $x_1=2$ is

$\Delta f(x_0)=f(2)-f(1)=\ln(2)-\ln(1)=\ln(2)=0.69315$

Ordinary Differences

- The second forward difference $\Delta^2 f(x_i)$ is defined as follows:

$$\Delta^2 f(x_i) = \Delta f(x_{i+1}) - \Delta f(x_i)$$

- Consequently the second divided difference expressed in terms of the ordinary difference is:

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_{i+1}, x_i]}{x_{i+2} - x_i} =$$

$$= \frac{1}{2h}\left[\frac{\Delta f(x_{i+1})}{h} - \frac{\Delta f(x_i)}{h}\right] = \frac{\Delta^2 f(x_i)}{2h^2}$$

Ordinary Differences

- The (k+1)st forward difference $\Delta^{k+1} f(x_i)$ is defined as follows:

$$\Delta^{k+1} f(x_i) = \Delta^k f(x_{i+1}) - \Delta^k f(x_i)$$

- In general,

$$f[x_i, \ldots, x_{i+k}] = \frac{\Delta^k f(x_i)}{k!h^k}$$

- Computing ordinary differences is the same as computing divided differences – in a table.

Example

- Compute the ordinary differences table for

$$f(x) = 2x^3$$

for the points:

$x_0=0$, $x_1=0.5$, $x_2=1$, $x_3=1.5$, $x_4=2$, $x_5=2.5$

- Compute the divided differences table for the same function and the same points.
- Compare the two tables.

Example – Table of Ordinary Differences

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	0				
0.5	0.25	0.25			
1	2	1.75	1.5		
1.5	6.75	4.75	3.0	1.5	
2	16	9.25	4.5	1.5	0
2.5	31.25	15.25	6.0	1.5	0
3	54	22.75	7.5	1.5	0

Example – Table of Divided Differences

x	f(x)	Ist DD	IInd DD	IIIrd DD	IVth DD
0	0				
0.5	0.25	0.5			
1	2	3.5	3		
1.5	6.75	9.5	6	2	
2	16	18.5	9	2	0
2.5	31.25	30.5	12	2	0
3	54	45.5	15	2	0

Example – Remarks

- The IVth DD of f(x) are zero. That is because the IVth DD of f(x) is approximated by f''''(ξ) which is zero.
- Ist DD = Ist difference/h = 2 (Ist difference)
- IInd DD = IInd difference/h(2h)= 2 (IInd difference)
- IIIrd DD = IIIrd difference/(h(2h)(3h))= 4/3(IIIrd difference)

Interpolating with Ordinary Differences

- An interpolation polynomial of degree n can be written in terms of ordinary differences.
- The independent variable in this polynomial is typically not x but s:

$$s = \frac{x - x_0}{h}$$

- Newton's forward difference formula is given by:

$$P_n(s) = f(x_0) + s\Delta f(x_0) + \frac{s(s-1)}{2!}\Delta^2 f(x_0) + \ldots$$

$$\ldots + \frac{s(s-1)\ldots(s-n+1)}{n!}\Delta^n f(x_0)$$

Example:

- Given the table of x_i and $f(x_i)$:

x	0	0.2	0.4	0.6	0.8	1.0	1.2
f(x)	0	0.203	0.423	0.684	1.03	1.557	2.572

- Compute the forward differences to order four.
- Find f(0.73) from a cubic interpolating polynomial.

Example – Solution

- We complete the table

x	f(x)	Ist diff	IInd diff	IIIrd diff	IVth diff
0	0				
0.2	0.203	0.203			
0.4	0.423	0.22	0.017		
0.6	0.684	0.261	0.041	0.024	
0.8	1.03	0.346	0.085	0.044	0.2
1.0	1.557	0.527	0.181	0.096	0.052
1.2	2.572	1.015	0.488	0.307	0.211

- 0.73

Example – Solution

- Since 0.73 falls between 0.6 and 0.8 and we need 4 point to obtain a cubic polynomial, we use the closest points to 0.73:

x_0	x_1	x_2	x_3
0.4	0.6	0.8	1

- We take the appropriate subtable:

x	f(x)	Ist diff	IInd diff	IIIrd diff
0.4	0.423			
0.6	0.684	0.261		
0.8	1.03	0.346	0.085	
1.0	1.557	0.527	0.181	0.096

Example – Solution

- We obtain the polynomial:

$$P_3(s) = 0.423 + 0.261s + 0.085\frac{s(s-1)}{2} + $$
$$+ 0.096\frac{s(s-1)(s-2)}{6}$$

- Since x=0.73, then

  $s=(x-x_0)/h=(0.73-0.4)/0.2=1.65$

  $P(1.65) = 0.893$

- Note: The function f(x)=tan(x). So f(0.73)=0.895. Thus the actual error of the approximation is 0.002.

Backward Differences

- As before, we can rearrange the points and define backward differences:

$$x_n \quad x_{n-1} \quad \ldots \quad x_1 \quad x_0$$

- **Definition**: The first backward difference at x_i is defined as follows:

$$\nabla f(x_i) = f(x_i) - f(x_{i-1})$$

- **Note:**

$$\nabla f(x_i) = \Delta f(x_{i-1})$$

Backward Differences

- **Definition**: The kth backward difference at the point x_i is defined as follows:

$$\nabla^k f(x_i) = \nabla^{k-1} f(x_i) - \nabla^{k-1} f(x_{i-1})$$

- **Definition**: Newton's backward difference formula is given by

$$P_n(s) = f(x_n) + s\nabla f(x_n) + \frac{s(s+1)}{2!}\nabla^2 f(x_n) + \ldots$$

$$\ldots + \frac{s(s+1)\ldots(s+n-1)}{n!}\nabla^n f(x_n)$$

where $s=(x-x_n)/h$.

Example

- Given the data:

x	−0.75	−0.5	−0.25	0
f(x)	−0.0718125	−0.02475	0.3349375	1.101

- Construct the forward difference table.
- Use Newton's forward difference formula to construct the interpolating polynomial of degree 3.
- Use Newton's backward difference formula to construct the interpolating polynomial of degree 3.
- Use either polynomial to approximate f(−1/3).

Example – Solution

- We construct the forward difference table:

x	f(x)	Ist diff	IInd diff	IIIrd diff
−0.75	−0.0718125			
−0.5	−0.02475	0.0470625		
−0.25	0.3349375	0.3596875	0.312625	
0	1.101	0.7660625	0.406375	0.09375

- The forward difference polynomial is

$$P_3(s) = -0.0718125 + 0.0470625s + 0.312625\frac{s(s-1)}{2!} +$$

$$+0.09375\frac{s(s-1)(s-2)}{3!}$$

Example – Solution

- The backward difference table is exactly the same as the forward difference table

x	f(x)	Ist diff	IInd diff	IIIrd diff
−0.75	−0.0718125			
−0.5	−0.02475	0.0470625		
−0.25	0.3349375	0.3596875	0.312625	
0	1.101	0.7660625	0.406375	0.09375

- The backward difference polynomial is:

$$P_3(s) = 1.101 + 0.7660625s + 0.406375\frac{s(s+1)}{2!} + $$

$$+0.09375\frac{s(s+1)(s+2)}{3!}$$

Example – Solution

- We have to use either polynomial to estimate $f(-1/3)$.
- If we use the backward polynomial,

  $s=(x-x_n)/h=x/h = -4/3$
- We compute $P_3(-4/3) \approx 0.1745185$