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Kolmogorov's differential equations and positive semigroups on first moment sequence spaces

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Abstract. Spatially implicit metapopulation models with discrete patch-size structure and host-macroparasite models which distinguish hosts by their parasite loads lead to infinite systems of ordinary differential equations. In several papers, a this-related theory will be developed in sufficient generality to cover these applications. In this paper the linear foundations are laid. They are of own interest as they apply to continuous-time population growth processes (Markov chains). Conditions are derived that the solutions of an infinite linear system of differential equations, known as Kolmogorov's differential equations, induce a C_0-semigroup on an appropriate sequence space allowing for first moments. We derive estimates for the growth bound and the essential growth bound and study the asymptotic behavior. Our results will be illustrated for birth and death processes with immigration and catastrophes.

1. Introduction

Spatially implicit metapopulation models with discrete patch-size structure [2,4, 6,29,31] and host-macroparasite models which distinguish hosts by their parasite loads [5,10,16,17,27,28,35,36] lead to infinite systems of ordinary differential equations (or partial differential equations if host-age is included),

$$\begin{aligned} w' &= f(t, w, x), \qquad (1) \\ x_j' &= \sum_{k=0}^{\infty} \alpha_{jk} x_k + g_j(t, w, x), \qquad j = 0, 1, 2, \ldots . \end{aligned}$$

where $x(t)$ is the sequence of functions $(x_j(t))_{j=0}^{\infty}$. The connection between these types of models is not incidental as a macroparasite population is a metapopulation with the hosts being the patches. In the equations above, x_j denotes the number of patches with j residents (number of hosts with j parasites) and w the average number of migrating individuals, or wanderers (average number of free-living parasites). The coefficients α_{jk} describe the transition from patches with k residents (hosts with k parasites) to patches with j residents (hosts with j parasites) due to

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deaths, births and emigration of residents (release of parasites). They have the properties typical for infinite transition matrices in stochastic processes with continuous time and discrete state (continuous-time birth and death chains, e.g., see [1,12,13] and the references therein). Typically the sequence of diagonal coefficients α_{kk} is unbounded. The function f gives the rate of change of the number of dispersers (free-living parasites) due to patch emigration, immigration and disperser death. The functions g_j describe the rate of change of the numbers of patches with j residents (hosts with j parasites) due to the immigration of dispersers (invasion of parasites). One possible approach to these equations, chosen by Karl–Peter Hadeler and collaborators [10,16,17,28], derives and analyzes partial differential equations for the moment generating functions of $x_j(t)$. This even works for infinite systems of partial differential equations and yields impressive and illuminating results, but requires the matrix α_{jk} to be essentially tridiagonal and α_{jk} to depend on j and k in a rather restricted way. It is our aim to develop a theory of semiflows on an appropriate sequence space which works without these restrictions [29,30] and in particular to establish conditions for the solution semiflow to be dissipative [18], have a compact attractor for bounded sets [18,39], and be uniformly persistent [19, 42,4,44]. We choose a somewhat more abstract approach than the ones in [2] and [4] from which we have received much inspiration in order to include a variety of models (in [29] we assume that only juveniles migrate) and to include state transitions which are not of nearest-neighbor type.

The biological interpretation gives us guidance how to choose the appropriate state space [2]. Assuming that meaningful solutions are non-negative, the number of patches (hosts) is given by $\sum_{j=0}^{\infty} x_j$ and the number of residents (in-host parasites) by $\sum_{j=1}^{\infty} jx_j$. In a population growth process, $\sum_{j=0}^{\infty} x_j = 1$ and $\sum_{j=1}^{\infty} jx_j$ is the expected population size. We recall the standard sequence space notation

$$\ell^1 = \left\{ (x_j)_{j=0}^{\infty};\ x_j \in \mathbb{R}, \sum_{j=0}^{\infty} |x_j| < \infty \right\} \tag{2}$$

with norm

$$\|x\| = \sum_{j=0}^{\infty} |x_j|, \qquad x = (x_j)_{j=0}^{\infty}, \tag{3}$$

and introduce the first-moment space [2]

$$\ell^{11} = \left\{ (x_j)_{j=0}^{\infty};\ x_j \in \mathbb{R}, \sum_{j=0}^{\infty} j|x_j| < \infty \right\}. \tag{4}$$

As norm on ℓ^{11} we choose

$$\|x\|_1 = \sum_{j=0}^{\infty} (1+j)|x_j|, \qquad x = (x_j)_{j=0}^{\infty}. \tag{5}$$

Other, equivalent, choices are possible, of course. In [29,30], we treat (1) as a semilinear operator differential equation

$$w' = f(t, w, x), \qquad x' = A_1 x + g(t, w, x) \tag{6}$$

on the non-negative cone of the Banach space $\mathbb{R} \times \ell^{11}$ where A_1 is the infinitesimal generator of a C_0-semigroup on ℓ^{11} and the functions f and $g(\sqcup) = \big(g_j(\sqcup)\big)_{j=0}^{\infty}$ are locally Lipschitz continuous. In this paper, we will concentrate on the linear problem which is of its own interest and a classic known as Kolmogorov's differential equations [26]; among other things, it is associated with continuous-time Markov chains which describe population growth (Sect. 2). See [12, XVII.9, 13, XIV.7, 24, Sect. 23.10–23.12, 25, 37, 38] and, for more recent references, [3,45]. We first construct a C_0-semigroup on ℓ^1 (Sect. 3). The approximation we use, as natural as it is, seems to be different from the ones used before [24, Sect. 23.10–23.12, 25, 46], [25,46] and is fundamental not only for this paper, but also for the analysis of (6) in [29,30]. Under an additional assumption, this semigroup leaves ℓ^{11} invariant and its restriction to ℓ^{11} is a C_0-semigroup as well (Sect. 4). Interestingly enough, the role of the first moment space ℓ^{11} suggests a new condition for substochastic semigroups to be stochastic [45]. In Section 5, we derive estimates for the growth bound of the semigroup and conditions for the semigroup to converge to 0 as time tends to infinity. Using a sequential characterization of the Kuratowski measure of noncompactness, we find estimates for the essential growth bound and conditions for convergence towards a non-zero steady state (Sect. 7). These results are illustrated in Sect. 8 for birth and death processes with immigration and catastrophes.

2. The Markov chain connection

We write $\mathbb{Z}_+$ for the set of non-negative integers and $\mathbb{N}$ for the natural numbers starting at 1, $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. We study the linear infinite system

$$x_j' = \sum_{k=0}^{\infty} \alpha_{jk} x_k, \quad j \in \mathbb{Z}_+, \tag{7}$$

known as *Kolmogorov's differential equations* [26,12,25,24]. In a Markov chain modeling population growth, $x = (x_j)$ is the probability distribution of the population size, i.e., $x_j(t)$ is interpreted as the probability that the population size is j at time t,

$$x_j(t) = \text{prob}\{X(t) = j\}.$$

Here $X(t)$ is the random variable describing the population size at time t. The coefficients α_{jk} are the transition rates between the population sizes. To see this, we introduce the transition probabilities

$$p_{jk}(t, s) = \text{prob}\{X(t) = j | X(s) = k\}$$

which are the conditional probabilities that the population size is j at time t provided it has been k at time $s < t$. (Cf. [13, XVII.9, 24, 23.10–23.12].) Obviously

$$p_{jk}(t,t) = \begin{Bmatrix} 1, & j = k, \\ 0, & j \neq k \end{Bmatrix} =: \delta_{jk} \tag{8}$$

with the usual Kronecker symbols δ_{jk}. Then the transition rates α_{jk} satisfy

$$\alpha_{jk} = \lim_{h \to 0+} \frac{p_{jk}(t+h,t) - \delta_{jk}}{h},$$

provided that these limits exist and do not depend on t. Since $0 \leq p_{jk} \leq 1$, this implies that $\alpha_{jk} \geq 0$ if $j \neq k$ and $\alpha_{jj} \leq 0$. Further

$$\sum_{j=0}^{\infty} p_{jk}(t+h,t) = \text{ prob } \{X(t+h) \in \mathbb{Z}_+ | X(t) = k\} = 1.$$

Formally interchanging the limit and the series suggests that $\sum_{j=0}^{\infty} \alpha_{jk} = 0$ for all $k \in \mathbb{Z}_+$. An infinite matrix with these properties is often called a *Kolmogorov* matrix [24, Sect. 23.12].

Notice that $\sum_{j=1}^{\infty} j p_{jk}(t,s)$ is the expected population size at time t under the condition that the population has size k at time s. Again, formally interchanging the limit and the series, we see that $\sum_{j=1}^{\infty} j\alpha_{jk}$ is the expected population growth rate at population size k.

Equation (7) follows from the Markov property:

$$\begin{aligned} x_j(t+h) &= \text{prob}\{X(t+h) = j\} \\ &= \sum_{k=0}^{\infty} \text{prob}\{X(t+h) = j | X(t) = k\} \, \text{prob}\{X(t) = k\} \\ &= \sum_{k=0}^{\infty} p_{jk}(t+h,t) x_k(t). \end{aligned} \tag{9}$$

So

$$x_j(t+h) - x_j(t) = \sum_{k=0}^{\infty} (p_{jk}(t+h,t) - \delta_{jk}) x_k(t).$$

The differential equations (7) follow by dividing by h and taking the limit as $h \to 0+$. Equation (9) also shows that the solution of (7) is given by

$$x_j(t) = \sum_{k=0}^{\infty} p_{jk}(t,0) x_k(0). \tag{10}$$

The right hand side of this equation defines the semigroup generated by the infinite matrix (α_{jk}). In this section, we have considered the transition probabilities p_{jk} as given and found the transition rates α_{jk} as their derivatives. In the following we will reverse the process and start from the transition rates α_{jk}. Our considerations suggest that meaningful assumptions should relate to $\sum_{j=0}^{\infty} \alpha_{jk}$ and $\sum_{j=1}^{\infty} j\alpha_{jk}$, $k \in \mathbb{Z}_+$.

3. Construction of the semigroup

The considerations in the preceding section motivate the following assumptions for the transition rates α_{jk}.

Assumption 1. *(a) For all $j, k \in \mathbb{Z}_+$, $\alpha_{jk} \geq 0$ if $j \neq k$, $\alpha_{jj} \leq 0$.*

(b) $\alpha^\diamond := \sup_{k=0}^{\infty} \sum_{j=0}^{\infty} \alpha_{jk} < \infty$.

Notice that the sequence (α_{jj}) may be unbounded and is so in many applications. Recall the standard sequence space ℓ^1 in (2); ℓ^1_+ denotes the cone of non-negative sequences in ℓ^1. Rather than Kolmogorov's differential equations (7) we consider the essentially finite linear system

$$\frac{\mathrm{d}}{\mathrm{dt}} x_j^{[n]} = \begin{cases} \sum_{k=0}^{n} \alpha_{jk} x_k^{[n]}, & j = 0, \dots, n, \\ \alpha_{jj} x_j^{[n]}, & j > n. \end{cases} \tag{11}$$

For given initial data, this system has a unique solution. It corresponds to (7) with the infinite matrix $(\alpha_{jk})_{j,k=0}^{\infty}$ being replaced by the infinite matrix $(\alpha_{jk}^{[n]})_{j,k=0}^{\infty}$, $n \in \mathbb{Z}_+$, given by

$$\alpha_{jk}^{[n]} = \left\{ \begin{array}{ll} \alpha_{jk}; & j, k \leq n \\ \alpha_{jj}; & j = k > n \\ 0; & \text{otherwise} \end{array} \right\}, \qquad j, k, n \in \mathbb{Z}_+. \tag{12}$$

Theorem 1. *Let $S^{[n]}(t)\breve{x} = x^{[n]}(t)$ denote the unique solutions of system (11) with initial data $\breve{x} = (\breve{x}_j)$. Then $S^{[n]}$, $n \in \mathbb{N}$, is a sequence of C_0-semigroups on ℓ^1. There exists a C_0-semigroup S on ℓ^1 such that $S^{[n]}(t)\breve{x} \to S(t)\breve{x}$ in ℓ^1 for every $\breve{x} \in \ell^1$, $t \geq 0$. If $\breve{x} \in \ell^1_+$, $S^{[n]}(t)\breve{x} \in \ell^1_+$, $S(t)\breve{x} \in \ell^1_+$, and the convergence of $S^{[n]}(t)\breve{x}$ to $S(t)\breve{x}$ as $n \to \infty$ is monotone increasing. The following estimates hold,*

$$\|S^{[n]}(t)\| \leq \|S(t)\| \leq e^{\alpha^\diamond t}, \quad t \geq 0.$$

Proof. One readily checks that the infinite matrices $(\alpha_{jk}^{[n]})$ satisfy Assumption 1. The system (7) with $(\alpha_{jk}^{[n]})$ instead of (α_{jk}) is identical with (11) and basically is a finite linear system of ordinary differential equations with a quasi-positive matrix. For every $\breve{x} \in \ell^1_+$ and $n \in \mathbb{N}$, there exists a unique non-negative solution $x^{[n]} \in \ell^1_+$ with $x^{[n]}(0) = \breve{x}$. In fact, $(x_1^{[n]}, \dots, x_n^{[n]})$ is the unique solution of a finite system of ordinary differential equations while

$$x_j^{[n]}(t) = \breve{x}_j \mathrm{e}^{\alpha_{jj} t}, \quad j > n. \tag{13}$$

We define $S^{[n]}(t)\breve{x} = x^{[n]}(t)$. It is not difficult to see that, for each $n \in \mathbb{N}$, this definition provides a family of bounded linear operators $\{S^{[n]}(t); t \geq 0\}$ on ℓ^1 which is strongly continuous in $t \geq 0$. Uniqueness of solutions implies that, for each $n \in \mathbb{N}$, we obtain a C_0-semigroup $S^{[n]}$.

The following estimates hold for $m, n \in \mathbb{N}$, $m \le n$:

$$\alpha_{jk}^{[m]} \le \alpha_{jk}^{[n]} \le \alpha_{jk} \quad \forall j, k \in \mathbb{N}. \tag{14}$$

Consider $\breve{x} \in \ell_+^1$. We claim that

$$0 \le x_j^{[m]}(t) \le x_j^{[n]}(t) \quad \forall j \in \mathbb{N}, t \ge 0. \tag{15}$$

Indeed: For $j > n > m$, $x_j^{[m]}(t) = \breve{x}_j \mathrm{e}^{\alpha_{jj} t} = x_j^{[n]}(t)$. If $m < j \le n$,

$$\frac{\mathrm{d}}{\mathrm{dt}} x_j^{[n]} = \sum_{k=0}^{n} \alpha_{jk} x_k^{[n]} \ge \alpha_{jj} x_j^{[n]}.$$

We integrate this inequality and use (13),

$$x_j^{[n]}(t) \ge \breve{x}_j \mathrm{e}^{\alpha_{jj} t} = x_j^{[m]}(t).$$

For $j = 0, \dots, m$,

$$\frac{\mathrm{d}}{\mathrm{d}t} x_j^{[n]} = \sum_{k=0}^{n} \alpha_{jk} x_k^{[n]} \ge \sum_{k=0}^{m} \alpha_{jk} x_k^{[n]}.$$

Recall that we have equality for $x^{[m]}$. Since the matrix $(\alpha_{jk})_{0 \le j,k \le m}$ is quasi-positive, it follows that $x_j^{[m]}(t) \le x_j^{[n]}(t)$ for $j = 0, \dots, m$ [41, Theorem B.1]. We add the equations in (11) and, by Assumption 1,

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{j=0}^{n} x_j^{[n]} = \sum_{j=0}^{n} \left(\sum_{k=0}^{n} \alpha_{jk} x_k^{[n]} \right) \le \sum_{k=0}^{n} \left(\sum_{j=0}^{\infty} \alpha_{jk} \right) x_k^{[n]} \le \alpha^\diamond \sum_{k=0}^{n} x_k^{[n]}.$$

We integrate this differential inequality,

$$\sum_{j=0}^{n} x_j^{[n]}(t) \le \mathrm{e}^{\alpha^\diamond t} \|\breve{x}\|. \tag{16}$$

This inequality together with the inequality (15) implies that

$$x_j^{[n]}(t) \to x_j(t), \qquad n \to \infty, \;\; j \in \mathbb{N}, \;\; t \ge 0,$$

where x_j are Borel measurable functions on $[0, \infty)$. By (16),

$$\sum_{j=0}^{\infty} x_j(t) \le \mathrm{e}^{\alpha^\diamond t} \|\breve{x}\|.$$

By Beppo Levi's theorem of monotone convergence, $\|x(t) - x^{[n]}(t)\| = \sum_{j=0}^{\infty} \left(x_j(t) - x_j^{[n]}(t) \right) \to 0$ as $n \to \infty$, pointwise in $t \ge 0$. As pointwise limit of continuous functions, x is strongly measurable in $t \ge 0$.

If $\breve{x} \in \ell^1$, we can find a representation $\breve{x} = \breve{x}_+ - \breve{x}_-$ and $\|\breve{x}\| = \|\breve{x}_+\| + \|\breve{x}_-\|$. We perform the previous procedure separately for $\breve{x}_+$ and $\breve{x}_-$ and find that the following limits exist and define strongly Borel measurable functions:

$$\lim_{n\to\infty} S^{[n]}(t)\breve{x}_+ - \lim_{n\to\infty} S^{[n]}(t)\breve{x}_- = \lim_{n\to\infty} S^{[n]}(t)\breve{x} =: S(t)\breve{x}.$$

Moreover

$$\begin{aligned}\|S^{[n]}(t)\breve{x}\| &= \|S^{[n]}(t)\breve{x}_+ - S^{[n]}(t)\breve{x}_-\| \\ &\le \|S^{[n]}(t)\breve{x}_+\| + \|S^{[n]}(t)\breve{x}_-\| \le \|S(t)\breve{x}_+\| + \|S(t)\breve{x}_-\| \\ &\le e^{\alpha^\diamond t}(\|\breve{x}_+\| + \|\breve{x}_-\|) = e^{\alpha^\diamond t}\|\breve{x}\|,\end{aligned}$$

and so

$$\|S^{[n]}(t)\breve{x}\| \le \|S(t)\breve{x}\| \le \|\breve{x}\| e^{\alpha^\diamond t}.$$

By the triangle inequality,

$$\begin{aligned}&\left\| S(t)S(s)\breve{x} - S^{[n]}(t)S^{[n]}(s)\breve{x} \right\| \\ &\le \left\| \left[S(t) - S^{[n]}(t) \right] S(s)\breve{x} \right\| + \left\| S^{[n]}(t) \left[S(s)\breve{x} - S^{[n]}(s)\breve{x} \right] \right\| \\ &\le \left\| [S(t) - S^{[n]}(t)]S(s)\breve{x} \right\| + e^{\alpha^\diamond t} \left\| S(s)\breve{x} - S^{[n]}(s)\breve{x} \right\| \\ &\longrightarrow 0, \qquad n \to \infty.\end{aligned}$$

This implies that S is a semigroup which is strongly Borel measurable in $t \ge 0$. By [24, Theorem 10.2.3], $S(t)$ is strongly continuous in $t > 0$. To see the strong continuity at $t = 0$ we proceed as in the proof [46, 1.4 (a)]. Pick any of the C_0-semigroups $S^{[n]}$ and let $x \in \ell^1_+$. Then

$$\|S(t)x - x\| \le \|S(t)x - S^{[n]}(t)x\| + \|S^{[n]}(t)x - x\|.$$

Since $S(t)x \ge S^{[n]}(t)x$,

$$\begin{aligned}\|S(t)x - x\| &\le \|S(t)x\| - \|S^{[n]}(t)x\| + \|S^{[n]}(t)x - x\| \\ &\le e^{\alpha^\diamond t}\|x\| - \|S^{[n]}(t)x\| + \|S^{[n]}(t)x - x\| \to 0, \quad t \to 0.\end{aligned}$$

□

We see from (13) that the C_0-semigroup $S^{[n]}$ has the infinitesimal generator $A^{[n]}$,

$$\begin{aligned}A^{[n]}x &= \left(\sum_{k=0}^{\infty} \alpha_{jk}^{[n]} x_k \right)_{j=0}^{\infty}, \qquad x = (x_k)_{k=0}^{\infty}, \\ D(A^{[n]}) &= \left\{ x \in \ell^1; \sum_{k=0}^{\infty} |\alpha_{kk}|\,|x_k| < \infty \right\} =: D_0.\end{aligned} \tag{17}$$

The following operator, also defined on D_0, will be useful,

$$Ax = \left(\sum_{k=0}^{\infty} \alpha_{jk} x_k \right)_{j=0}^{\infty}, \qquad x \in D_0. \tag{18}$$

Lemma 1. *D_0 is dense in ℓ^1, $A : D_0 \to \ell^1$ is well-defined and linear and has the following properties:*

(a) the estimates and equality

$$\|(\lambda - A)x\| \geq (\lambda - \alpha^\diamond)\|x\| \qquad \forall x \in D_0, \lambda \in \mathbb{R},$$

$$\sum_{j=0}^{\infty}(Ax)_j = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{\infty}\alpha_{jk}\right)x_k \leq \alpha^\diamond\|x\| \qquad \forall x \in D_0 \cap \ell^1_+,$$

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|\alpha_{jk}||x_k| \leq \alpha^\diamond\|x\| + 2\sum_{j=0}^{\infty}|\alpha_{jj}|\,|x_j| \qquad \forall x \in D_0.$$

(b) For $x \in D_0$, $A^{[n]}x \to Ax$ as $n \to \infty$.

Proof. Let $x \in D_0$. Then

$$\begin{aligned}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|\alpha_{jk}||x_k| &= \sum_{j=0}^{\infty}\sum_{k=0,k\neq j}^{\infty}\alpha_{jk}|x_k| + \sum_{j=0}^{\infty}|\alpha_{jj}|\,|x_j| \\ &= \sum_{k=0}^{\infty}\left(\sum_{j=0,k\neq j}^{\infty}\alpha_{jk}\right)|x_k| + \sum_{j=0}^{\infty}|\alpha_{jj}||x_j| \\ &= \sum_{k=0}^{\infty}\left(\sum_{j=0}^{\infty}\alpha_{jk}\right)|x_k| + 2\sum_{j=0}^{\infty}|\alpha_{jj}||x_j| \\ &\leq \alpha^\diamond\|x\| + 2\sum_{j=0}^{\infty}|\alpha_{jj}||x_j|.\end{aligned}$$

This estimate implies that Ax is well-defined for $x \in D_0$. Obviously A is linear and D_0 dense as it contains each sequence all terms of which are zero except one.

(a) For $x \in D_0$, choose $x^* \in \ell^\infty$ as

$$x_j^* = \begin{cases} 1, & x_j \geq 0, \\ -1, & x_j < 0. \end{cases}$$

Then $\langle x, x^*\rangle = \|x\|$ and

$$\langle(\lambda - A)x, x^*\rangle = \lambda\|x\| - \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\alpha_{jk}x_k x_j^* \geq \lambda\|x\| - \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\alpha_{jk}|x_k|.$$

Since $\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|\alpha_{jk}||x_k| < \infty$, we can change the order of summation,

$$\langle(\lambda - A)x, x^*\rangle \geq \lambda\|x\| - \sum_{k=0}^{\infty}\left(\sum_{j=0}^{\infty}\alpha_{jk}\right)|x_k| \geq \lambda\|x\| - \alpha^\diamond\|x\|.$$

By the same token, for all $x \in D_0 \cap \ell^1_+$,

$$\sum_{j=0}^{\infty}(Ax)_j = \sum_{j=0}^{\infty}\left(\sum_{k=0}^{\infty}\alpha_{jk}x_k\right) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{\infty}\alpha_{jk}\right)x_k \le \alpha^\diamond \|x\|.$$

(b) Let $x \in D_0$. By the definitions of A in (18) and $A^{[n]}$ in (12) and (17),

$$(Ax)_j - \left(A^{[n]}x\right)_j = \begin{cases} \displaystyle\sum_{k=n+1}^{\infty} \alpha_{jk}x_k; & j = 1, \dots, n; \\ \displaystyle\sum_{k=0, k\neq j}^{\infty} \alpha_{jk}x_k; & j = n+1, \dots. \end{cases}$$

Since all occurring α_{jk} are non-negative,

$$\begin{aligned} \left\| Ax - A^{[n]}x \right\| &\le \sum_{j=0}^{n}\sum_{k=n+1}^{\infty} \alpha_{jk}|x_k| + \sum_{j=n+1}^{\infty}\sum_{k=0,k\neq j}^{\infty} \alpha_{jk}|x_k| \\ &= \sum_{k=n+1}^{\infty} |x_k| \sum_{j=0, j\neq k}^{\infty} \alpha_{jk} + \sum_{k=0}^{n} |x_k| \sum_{j=n+1}^{\infty} \alpha_{jk}. \end{aligned}$$

By Assumption 1, the first term can be estimated by

$$\sum_{k=n+1}^{\infty} |x_k|(\alpha^\diamond + |\alpha_{kk}|)$$

and converges to 0 as $n \to \infty$ because $x \in D_0$. The second term converges to 0 by Lebesgue's dominated convergence theorem because $x \in \ell^1$ and $\sum_{j=n+1}^{\infty} \alpha_{jk} \to 0$ as $n \to \infty$ by Assumption 1. □

The following result is derived in [45] which also contains a more semigroup-oriented proof of Theorem 1 than the one given here. It is also proved in [25,24, 46] based on other approximation procedures.

Proposition 1. *The infinitesimal generator of the C_0-semigroup S in Theorem 1 is an extension of A. Further S is minimal with respect to this property: If $\tilde{S}$ is a positive C_0-semigroup on ℓ^1 whose generator extends A, then $S(t)x \le \tilde{S}(t)x$ for all $x \in \ell^1_+$, $t \ge 0$.*

4. The semigroup on its natural state space

In view of our applications, the state space of main interest is the first moment space

$$\ell^{11} = \left\{ x \in \ell^1; \sum_{j=1}^{\infty} j|x_j| < \infty \right\}$$

with norm $\|x\|_1 = \|x\| + \sum_{j=1}^{\infty} j|x_j|$.

Assumption 2. *There exist constants $c_0, c_1 > 0, \epsilon > 0$ such that*

$$\sum_{j=1}^{\infty} j\alpha_{jk} \le c_0 + c_1 k - \epsilon|\alpha_{kk}| \qquad \forall k \in \mathbb{Z}_+.$$

Since $\alpha_{jk} \ge 0$ if $j \ne k$, this assumption implies that the series converges. To appreciate this assumption, notice that $\sum_{j=1}^{\infty}(j/k)\alpha_{jk}$ is the expected per capita growth rate at population size k. So the assumption in particular states that these per capita growth rates are bounded. The term $-\epsilon|\alpha_{kk}|$ is difficult to interpret, but neither restricts the applicability of the assumption too much (Sect. 8) nor is of completely technical nature. When this term is present in Assumption 2, the generator of the semigroup S in Theorem 1 is the closure of the operator A in (18) [45], while, without it, this may not be the case [3, Theorem 7.11] [25] and solutions to (7) may not be uniquely determined by their initial data [37, Sect. 6].

We now turn to the state space ℓ^{11}.

Theorem 2. *Let the Assumptions 1 and 2 be satisfied. Then the following hold:*

(a) The C_0-semigroup S on ℓ^1 in Theorem 1 leaves ℓ^{11} invariant. Its restriction to ℓ^{11}, S_1, is a C_0-semigroup on ℓ^{11} which is generated by the part of A in ℓ^{11}, denoted by A_1, i.e., A_1 is the restriction of A to

$$D(A_1) = \{x \in \ell^{11} \cap D_0;\ Ax \in \ell^{11}\}.$$

Further $\|S_1(t)\|_1 \le \mathrm{e}^{\omega t}$ for all $t \ge 0$, with $\omega = \max\{c_1, \alpha^\diamond + c_0\}$.

(b) The semigroups $S^{[n]}$ in Theorem 1 leave ℓ^{11} invariant. Their restrictions to ℓ^{11}, $S_1^{[n]}$, are C_0-semigroup on ℓ^{11} and also satisfy the estimate $\|S_1^{[n]}(t)\|_1 \le \mathrm{e}^{\omega t}$ for all $t \ge 0$. The domain of their infinitesimal generators, $A_1^{[n]}$, are

$$D\left(A_1^{[n]}\right) = \left\{x \in \ell^{11};\ \sum_{j=1}^{\infty} j|\alpha_{jj}||x_j| < \infty\right\} =: D_1.$$

Finally, for all $\breve{x} \in \ell^{11}$, $S_1^{[n]}(t)\breve{x} \to S_1(t)\breve{x}$ in ℓ^{11}, with the convergence being uniform in bounded intervals in $\mathbb{R}_+$.

Proof. We first show that $\lambda - A_1$ has a bounded inverse and that $\|(\lambda - A_1)^{-1}\|_1 \le (\lambda - \omega)^{-1}$ for all $\lambda > \omega$. Let $\breve{x} \in \ell^{11}_+$. Revisit the construction in the proof of Theorem 1. We apply Laplace transforms to (11),

$$\lambda \hat{x}_j^{[n]} - \breve{x}_j = \sum_{k=0}^{n} \alpha_{jk}\hat{x}_k^{[n]}, \quad j = 0, \ldots, n,\ \lambda > \alpha^\diamond. \tag{19}$$

$\hat{x}_j^{[n]}(\lambda) = \int_0^\infty \mathrm{e}^{-\lambda t} x_j^{[n]}(t)dt$ denotes the Laplace transform of $x_j^{[n]}$ evaluated at λ. The Laplace transforms exist for $\lambda > \alpha^\diamond$ by (16). For convenience, we have dropped λ in the equation above. We take sums,

$$\lambda \sum_{j=0}^{n} \hat{x}_j^{[n]} - \sum_{j=0}^{n} \breve{x}_j = \sum_{j=0}^{n}\sum_{k=0}^{n} \alpha_{jk}\hat{x}_k^{[n]} \le \sum_{k=0}^{n}\left(\sum_{j=0}^{\infty}\alpha_{jk}\right)\hat{x}_k^{[n]}.$$

By Assumption 1,

$$\lambda \sum_{j=0}^{n} \hat{x}_j^{[n]} - \sum_{j=0}^{n} \breve{x}_j \leq \sum_{k=0}^{n} \alpha^\diamond \hat{x}_k^{[n]}.$$

We reorganize,

$$(\lambda - \alpha^\diamond) \sum_{j=0}^{n} \hat{x}_j^{[n]} \leq \sum_{j=0}^{n} \breve{x}_j.$$

We apply Beppo Levi's theorem of monotone convergence to the Laplace transforms,

$$\hat{x}_j^{[n]} \nearrow \hat{x}_j, \quad n \to \infty, \quad j \in Z_+,$$

where $x_j(t) = \lim_{n\to\infty} x_j^{[n]}(t)$. We take the limit as $n \to \infty$ in the previous inequality,

$$(\lambda - \alpha^\diamond)\|\hat{x}\| \leq \|\breve{x}\|. \tag{20}$$

Here $\hat{x} = (\hat{x}_j)$ is the Laplace transform of x. We take weighted sums of (19),

$$\begin{aligned}\lambda \sum_{j=1}^{n} j\hat{x}_j^{[n]} - \sum_{j=1}^{n} j\breve{x}_j &= \sum_{j=1}^{n} j \sum_{k=0}^{n} \alpha_{jk}\hat{x}_k^{[n]} \\ &\leq \sum_{k=0}^{n} \left(\sum_{j=1}^{\infty} j\alpha_{jk} \right) \hat{x}_k^{[n]}.\end{aligned}$$

By Assumption 2,

$$(\lambda - c_1) \sum_{j=1}^{n} j\hat{x}_j^{[n]} + \epsilon \sum_{k=0}^{n} |\alpha_{kk}|\hat{x}_k^{[n]} \leq \sum_{j=1}^{n} j\breve{x}_j + c_0 \sum_{k=0}^{\infty} \hat{x}_k^{[n]}.$$

Let $\lambda > \max\{c_1, \alpha^\diamond\}$ and take the limit $n \to \infty$,

$$(\lambda - c_1) \sum_{j=1}^{\infty} j\hat{x}_j + \epsilon \sum_{k=0}^{\infty} |\alpha_{kk}|\hat{x}_k \leq \sum_{j=1}^{\infty} j\breve{x}_j + c_0\|\hat{x}\|.$$

This implies that $\hat{x} \in D_0 \cap \ell^{11}$. We add this inequality and inequality (20),

$$\lambda\|\hat{x}\|_1 - (c_0 + \alpha^\diamond)\|\hat{x}\| - c_1 \sum_{j=1}^{\infty} j\hat{x}_j \leq \|\breve{x}\|_1.$$

Set $\omega = \max\{c_1, \alpha^\diamond + c_0\}$. Then

$$(\lambda - \omega)\|\hat{x}\|_1 \leq \|\breve{x}\|_1 \quad \forall \lambda > \omega.$$

We reorganize (19),

$$(\lambda - \alpha_{jj})\hat{x}_j^{[n]} = \check{x}_j + \sum_{k=0,k\neq j}^{n} \alpha_{jk}\hat{x}_k^{[n]}, \quad j = 0, \dots, n.$$

We apply Beppo Levi's theorem of monotone convergence,

$$(\lambda - \alpha_{jj})\hat{x}_j = \check{x}_j + \sum_{k=0,k\neq j}^{\infty} \alpha_{jk}\hat{x}_k, \quad j \in \mathbb{Z}_+.$$

We conclude that $A\hat{x} = \lambda\hat{x} - \check{x} \in \ell^{11}$, $\hat{x} \in D(A_1)$, and $(\lambda - A_1)\hat{x} = \check{x}$. If $\check{x} \in \ell^{11}$, we split $\check{x}$ in positive and negative part and obtain the same results. By Lemma 1, there is at most one solution $x \in D(A_1)$ of the equation of $(\lambda - A_1)x = \check{x}$ for $\lambda > \omega$, because it also solves $(\lambda - A)x = \check{x}$ and $\lambda - A$ is injective in ℓ^1 for $\lambda > \alpha^\diamond$ by Lemma 1. Summarizing, we have shown that $\lambda - A_1$ is invertible for $\lambda > \omega$ and $\|(\lambda - A_1)^{-1}\|_1 \leq (\lambda - \omega)^{-1}$ for all $\lambda > \omega$. In order to show that $D(A_1)$ is dense, we establish that $D(A_1)$ contains all the sequences all terms of which are 0 except one. Let $j \in \mathbb{Z}_+$ be fixed and $\mathrm{e}^j = (\delta_{jk})_{k=0}^\infty$ where δ_{jk} are the Kronecker symbols (8). Obviously $\mathrm{e}^j \in D_0 \cap \ell^{11}$. Further $A\mathrm{e}^j = (\alpha_{ij})_{i=0}^\infty$. By Assumption 2,

$$\sum_{i=1}^{\infty} i|(A\mathrm{e}^j)_i| = \sum_{i=1}^{\infty} i|\alpha_{ij}| = \sum_{i=1}^{\infty} i\alpha_{ij} + 2j|\alpha_{jj}| \leq c_0 + c_1 j + 2j|\alpha_{jj}| < \infty.$$

Hence $A\mathrm{e}^j \in \ell^{11}$ and $\mathrm{e}^j \in D(A_1)$.

(a) By the Hille–Yosida generation theorem [34, Sect.1.3: Corollary 3.8], A_1 is the infinitesimal generator of a C_0-semigroup $S_1(t)$ in ℓ^{11} satisfying $\|S_1(t)\|_1 \leq \mathrm{e}^{\omega t}$. In particular $(\lambda - A_1)^{-1}$ is the Laplace transform of S_1 for $\lambda > \omega$. The uniqueness properties of the Laplace transform imply that $S_1(t)\check{x} = x(t) = S(t)\check{x}$ for all $t \geq 0$. This implies that $S_1(t)$ is the restriction of $S(t)$ to ℓ^{11}.

(b) Since $\alpha_{jk}^{[n]} \leq \alpha_{jk}$, the Assumptions 1 and 2 also hold for the infinite matrix $\alpha_{jk}^{[n]}$ and the previous conclusions hold for $S^{[n]}$ as well. Let $\check{x} \in \ell_+^{11}$, $t \geq 0$. By Theorem 1, $[S^{[n]}(t)\check{x}]_j \nearrow [S_1(t)\check{x}]_j$ for each $j \in \mathbb{N}$. Since $S_1(t)\check{x} \in \ell^{11}$, Beppo Levi's theorem of monotone convergence implies that $S^{[n]}(t)\check{x} \to S_1(t)\check{x}$ in ℓ^{11}. Finally, since $\|S(t)\check{x} - S^{[n]}(t)\check{x}\|$ is a continuous function of t which converges pointwise and decreasing to 0, the convergence is uniform on every compact interval in $\mathcal{R}_+$ by Dini's lemma. □

An alternative proof can be found in [45]. The next result sheds some light on the relation between $D(A_1)$ and the set D_1 in Theorem 2 (b) which is the domain of the operators $A_1^{[n]}$.

Lemma 2. *Let the Assumptions 1 and 2 be satisfied. Then $D_1 \subseteq D(A_1)$ and for all $x \in D_1$,*

$$\sum_{k=0}^{\infty} \left(\sum_{j=1}^{\infty} j|\alpha_{jk}| \right) |x_k| < \infty.$$

Proof. By Assumption 1 and 2,

$$\sum_{j=1}^{\infty} j|\alpha_{jk}| \leq 2k|\alpha_{kk}| + c_0 + c_1 k \quad \forall k \in \mathbb{Z}_+.$$

For $x \in D_1$,

$$\sum_{k=0}^{\infty}\left(\sum_{j=1}^{\infty} j|\alpha_{jk}|\right)|x_k| \leq 2\sum_{k=1}^{\infty} k|\alpha_{kk}||x_k| + c_0\sum_{k=0}^{\infty}|x_k| + c_1\sum_{k=1}^{\infty} k|x_k| < \infty.$$

Obviously $D_1 \subseteq D_0$ and, for $x \in D_1$,

$$\sum_{j=1}^{\infty} j\left(\sum_{k=0}^{\infty}|\alpha_{jk}||x_k|\right) = \sum_{k=0}^{\infty}\left(\sum_{j=1}^{\infty} j|\alpha_{jk}|\right)|x_k| < \infty$$

which implies that $Ax \in \ell^{11}$ and $x \in D(A_1)$. $\square$

Remark 1. Let $\breve{x} \in \ell^{11}$, $x(t) = S_1(t)\breve{x}$. By standard semigroup theory [34, Chapter 1, Theorem 2.4], $\int_0^t x(s)ds \in D(A_1)$ and

$$x(t) = \breve{x} + A_1\int_0^t x(s)\mathrm{d}s. \tag{21}$$

Writing this equation term by term, we see that (7) is satisfied in an integral sense,

$$x_j(t) = \breve{x}_j + \sum_{k=0}^{\infty}\alpha_{jk}\int_0^t x_k(s)\mathrm{d}s, \quad t \geq 0. \tag{22}$$

Further, for $t \geq r \geq 0$,

$$\sum_{j=0}^{\infty}\left[x_j(t) - x_j(r)\right] = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{\infty}\alpha_{jk}\right)\int_r^t x_k(s)\mathrm{d}s. \tag{23}$$

Proof. For the proof of (23), it is sufficient to consider $\breve{x} \in \ell^{11}_+$. Let $x^{[n]}(t) = S_1^{[n]}(t)\breve{x}$ with the C_0-semigroups $S_1^{[n]}$ from Theorem 2. The same proof as in Theorem 1 provides

$$x_j^{[0]}(t) \leq x_j^{[m]}(t) \leq x_j^{[n]}(t) \leq x_j(t) \quad j \in \mathbb{Z}_+, t \geq 0, m, n \in \mathbb{N}, m \leq n.$$

Further

$$\alpha_{jk}^{[0]} \leq \alpha_{jk}^{[m]} \leq \alpha_{jk}^{[n]} \leq \alpha_{jk}.$$

Since $D(A_1^{[n]}) = D_1$, we can change the order of summation and obtain

$$\sum_{j=0}^{\infty}(x_j^{[n]}(t) - x_j^{[n]}(r)) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{\infty}\alpha_{jk}^{[n]}\right)\int_r^t x_k^{[n]}(s)\mathrm{d}s.$$

By construction, for each $k \in \mathbb{Z}_+$,

$$\alpha_{kk} \le \sum_{j=0}^{\infty} \alpha_{jk}^{[n]} \nearrow \sum_{j=0}^{\infty} \alpha_{jk} \le \alpha^{\diamond}, \quad n \to \infty.$$

Recall that $0 \le x_k^{[n]}(t) \nearrow x_k(t)$ for all $t \ge 0$, $k \in \mathbb{Z}_+$ as $n \to \infty$. So

$$\left| \left(\sum_{j=0}^{\infty} \alpha_{jk}^{[n]} \right) \int_r^t x_k^{[n]}(s) \mathrm{d}s \right| \le \max\{\alpha^{\diamond}, |\alpha_{kk}|\} \int_r^t x_k(s) \mathrm{d}s.$$

Since $\int_r^t x(s)\mathrm{d}s \in D(A_1) \subseteq D_0$, the sum of the right hand sides of this inequality over $k \in \mathbb{Z}_+$ is finite. Equation (23) now follows from the dominated convergence theorem. □

The following special case is important for Markov chains which describe population growth.

Lemma 3. *Let the Assumption 1 and Assumption 2 be satisfied and* $\sum_{j=0}^{\infty} \alpha_{jk} = 0$ *for all* $k \in \mathbb{Z}_+$*. Then the functional* v^* *defined by* $\langle x, v^* \rangle = \sum_{j=0}^{\infty} x_j$ *is an element in* $D(A_1^*)$ *and* $A_1^* v^* = 0$*. In particular, 0 is a spectral value of* A_1*. Further* $\langle S(t)\breve{x}, v^* \rangle = \langle \breve{x}, v^* \rangle$ *for all* $\breve{x} \in \ell^1$ *and* $\|S(t)\breve{x}\| = \|\breve{x}\|$ *for all* $\breve{x} \in \ell_+^{11}$ *and all* $t \ge 0$.

Proof. Let $x \in D(A_1)$. Then $x \in D_0$ and $\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |\alpha_{jk}||x_k| < \infty$ by Lemma 1. Since we can change the order of summation,

$$\langle A_1 x, v^* \rangle = \sum_{j=0}^{\infty} \left(\sum_{k=0}^{\infty} \alpha_{jk} x_k \right) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{\infty} \alpha_{jk} \right) x_k = 0.$$

By definition of a dual operator, $v^* \in D(A_1^*)$ and $A_1^* v^* = 0$. In particular 0 is an eigenvalue of A_1^*. Since A_1 and A_1^* have the same spectrum, 0 is a spectral value of A_1. Let $\breve{x} \in \ell_+^{11}$, $x(t) = S_1(t)\breve{x}$. We apply v^* to (21),

$$\langle x(t), v^* \rangle = \langle \breve{x}, v^* \rangle + \left\langle \int_0^t x(s)\mathrm{d}s, A_1^* v^* \right\rangle = \langle \breve{x}, v^* \rangle.$$

Since $x(t) \in \ell_+^{11}$, $\langle x(t), v^* \rangle = \|x(t)\|$. Hence $\|S_1(t)\breve{x}\| = \|\breve{x}\|$ for all $\breve{x} \in \ell_+^{11}$. Since ℓ_+^{11} is dense in ℓ_+^1 and ℓ^{11} dense in ℓ^1 and S an extension of S_1, the assertion follows. □

4.1. Semigroups on higher moment spaces

The m^{th} moment (sequence) space is defined as

$$\ell^{1m} = \left\{ x \in \ell^1; \sum_{j=1}^{\infty} j^m |x_j| < \infty \right\}$$

with norm $\|x\|_m = \|x\| + \sum_{j=1}^{\infty} j^m |x_j|$.

Assumption 3. There exists constants c_m and $\tilde{c}_m$ such that

$$\sum_{j=1}^{\infty} j^m \alpha_{jk} \leq c_m + \tilde{c}_m k^m \quad \forall k \in \mathbb{Z}_+.$$

Theorem 3. *Let the Assumptions 1 and 2, and, for a given $m \in \mathbb{N}$, $m > 1$, the Assumption 3 be satisfied. Then the C_0-semigroup S in Theorem 2 leaves ℓ^{1m} invariant and the restrictions $S_m(t)$ of $S(t)$ to ℓ^{1m} form a C_0-semigroup on ℓ^{1m}. S_m is generated by the part of A in ℓ^{1m} and $\|S_m(t)\|_m \leq \mathrm{e}^{\omega_m t}$ for all $t \geq 0$ with $\omega_m = \max\{\tilde{c}_m, \alpha^\diamond + c_m\}$.*

Proof. The proof is similar to the proof of Theorem 2 with j being replaced by j^m and the $\epsilon\alpha_{kk}$ terms being dropped. We can now use the result from Theorem 2 that $(\lambda - A_1)^{-1}$ maps ℓ^{11} and so ℓ^{1m} into D_0 which implies that A_m is one-to-one. □

5. Better estimates for the growth bound and population extinction

The growth bound of the semigroup S_1 is defined as

$$\omega := \omega(S_1) := \inf_{t>0} \frac{1}{t} \ln \|S_1(t)\|_1. \tag{24}$$

In Theorem 2 we derived the estimate $\omega(S_1) \leq \max\{c_1, \alpha^\diamond + c_0\}$ with the constants from Assumption 1 and 2. In this section we will derive better estimates under stronger assumptions. In particular, we will present conditions for the growth bound to be negative. For a Markov chain modeling population growth this means that the expected population size tends to 0 (exponentially fast) as time tends to infinity.

Assumption 4. There exist $m \in \mathbb{N}$ and $\epsilon_1 \in \mathbb{R}$ such that

$$\sum_{j=1}^{\infty} j\alpha_{jk} \leq -\epsilon_1 k \qquad \forall k > m.$$

$\sum_{j=1}^{\infty}(j/k)\alpha_{jk}$ can be interpreted as expected per capita population growth rate at population size k. So $-\epsilon_1$ can be interpreted as a bound of the expected per capita population growth rates at sufficiently large population sizes. Most of the time, but not always, we will use this assumption with $\epsilon_1 > 0$. Then the expected per capita growth rates are negative at large population sizes.

Lemma 4. *Let the Assumptions 1, 2, and 4 be satisfied. Then, for all $\breve{x} \in \ell^{11}$ and $x(t) = S_1(t)\breve{x}$,*

$$\sum_{j=1}^{\infty} j|x_j(t)| \leq \mathrm{e}^{-\epsilon_1 t} \sum_{j=1}^{\infty} j|\breve{x}_j| + \sum_{k=0}^{m} \xi_k \int_0^t \mathrm{e}^{-\epsilon_1(t-s)} |x_k(s)| \mathrm{d}s \qquad \forall t \geq 0,$$

with $\xi_k = \sum_{j=1}^{\infty} j\alpha_{jk} + \epsilon_1 k < \infty$.

Proof. Let $n > m$ with m from Assumption 4, $\breve{x} \in \ell^{11}_+$. Let $x^{[n]}(t) = S_1^{[n]}(t)\breve{x}$. By (11),

$$\frac{\mathrm{d}}{\mathrm{dt}} \sum_{j=1}^{n} j x_j^{[n]} = \sum_{k=0}^{n} \left(\sum_{j=1}^{n} j\alpha_{jk} \right) x_k^{[n]} \leq \sum_{k=0}^{n} \tilde{\xi}_k x_k^{[n]},$$

with $\tilde{\xi}_k = \sum_{j=1}^{\infty} j\alpha_{jk}$. In the last inequality we have used that $x_k^{[n]} \geq 0$ and $\alpha_{jk} \geq 0$ for $j \neq k$ (Assumption 1). By Assumption 4, $\tilde{\xi}_k \leq -\epsilon_1 k$ for all $k \geq m$, and

$$\frac{\mathrm{d}}{\mathrm{dt}} \sum_{j=1}^{n} j x_j^{[n]} \leq \sum_{k=0}^{m} \tilde{\xi}_k x_k^{[n]} - \epsilon_1 \sum_{k=m+1}^{n} k x_k^{[n]} = \sum_{k=0}^{m} \xi_k |x_k^{[n]}| - \epsilon_1 \sum_{j=1}^{n} j x_j^{[n]},$$

with $\xi_k = \tilde{\xi}_k + \epsilon_1 k$. We integrate this inequality,

$$\sum_{j=1}^{n} j x_j^{[n]}(t) \leq \mathrm{e}^{-\epsilon_1 t} \sum_{j=1}^{n} j\breve{x}_j + \sum_{k=0}^{m} \int_0^t \mathrm{e}^{-\epsilon_1(t-s)} \xi_k \left| x_k^{[n]}(s) \right| \mathrm{d}s.$$

By Theorem 2, we can take the limit for $n \to \infty$ and obtain the statement for $x(t) = S_1(t)\breve{x}$ with $\breve{x} \in \ell^{11}_+$. We use that every $\breve{x} \in \ell^{11}$ satisfies $\breve{x} = \breve{x}_+ - \breve{x}_-$ with $\breve{x}_\pm \in \ell^{11}_+$ and $\|\breve{x}\|_1 = \|\breve{x}_+\|_1 + \|\breve{x}_-\|_1$. Since $x^\pm(t) = S_1(t)\breve{x}_\pm$ and $|x_j(t)| = x_j^+(t) + x_j^-(t)$, the statement follows for every $\breve{x} \in \ell^{11}$. □

In the next theorem notice that $\sum_{j=1}^{\infty} \frac{j\alpha_{jk}}{k}$ can be interpreted as the expected per capita population growth rate at population size k.

Theorem 4. *Let the Assumptions 1 and 2 be satisfied. Then, for every $\omega \in \mathbb{R}$ with*

$$\omega \geq \sup_{k=0}^{\infty} \sum_{j=0}^{\infty} \alpha_{jk} \quad \textit{and} \quad \omega > \limsup_{k\to\infty} \sum_{j=1}^{\infty} \frac{j\alpha_{jk}}{k},$$

there exists some $M \geq 1$ such that $\|S_1(t)\|_1 \leq M\mathrm{e}^{\omega t}$ for all $t \geq 0$.

Proof. Let $x(t) = S_1(t)\breve{x}$. We can choose $\alpha^\diamond = \sup_{k=0}^{\infty} \sum_{j=0}^{\infty} \alpha_{jk}$ in Assumption 1. Further we choose some $\epsilon_1 \in \mathbb{R}$ such that $\omega > -\epsilon_1 > \limsup_{k\to\infty} \sum_{j=0}^{\infty} \frac{j\alpha_{jk}}{k}$ and $-\epsilon_1 \neq \alpha^\diamond$. Then Assumption 4 is satisfied with some $m \in \mathbb{N}$. By Theorem 1,

$\|x(t)\| \le e^{\alpha^\diamond t}\|\breve{x}\|$ for all $t \ge 0$. We substitute this inequality into the inequality of Lemma 4. With $\xi = \max_{k=0}^m \xi_k$,

$$\sum_{j=1}^{\infty} j|x_j(t)| \le e^{-\epsilon_1 t} \sum_{j=1}^{\infty} j|\breve{x}_j| + \xi \int_0^t e^{-\epsilon_1 s} e^{\alpha^\diamond (t-s)} \|\breve{x}\| ds \qquad \forall t \ge 0.$$

The statement follows from evaluating the integral and $\omega \ge \alpha^\diamond$, $\omega > -\epsilon_1$. □

Corollary 1. *Let the coefficients α_{jk}, $j, k \in \mathbb{Z}_+$, satisfy the following assumptions:*

(1) $\alpha_{jk} \ge 0$ for $j \ne k$, $\alpha_{jj} \le 0$ for all $j, k \in \mathbb{Z}_+$.
(2) $\sum_{j=0}^{\infty} \alpha_{jk} \le 0$ for all $k \in \mathbb{Z}_+$.
(3) There exist constants $c_0, c_1, \epsilon > 0$ such that

$$\sum_{j=1}^{\infty} j\alpha_{jk} \le c_0 + c_1 k - \epsilon|\alpha_{kk}| \qquad \forall k \in \mathbb{Z}_+.$$

(4) $\limsup_{k\to\infty} \frac{1}{k} \sum_{j=1}^{\infty} j\alpha_{jk} < 0$.

Then the semigroup $\{S_1(t)\}$ is bounded on ℓ^{11}.

Proof. The Assumptions of Theorem 4 are satisfied with $\omega = 0$. □

6. Exponential stability

Proposition 2. *Let the coefficients α_{jk}, $j, k \in \mathbb{Z}_+$, satisfy the following assumptions:*

(1) $\alpha_{jk} \ge 0$ for $j \ne k$, $\alpha_{jj} \le 0$.
(2) $\sum_{j=0}^{\infty} \alpha_{jk} \le 0$ for all $k \in \mathbb{Z}_+$.
(3) There exist constants $c_0, c_1, \epsilon > 0$ such that

$$\sum_{j=1}^{\infty} j\alpha_{jk} \le c_0 + c_1 k - \epsilon|\alpha_{kk}| \qquad \forall k \in \mathbb{Z}_+.$$

(4) $\alpha_{j0} = 0$ for all $j \in \mathbb{N}$.
(5) For all $k \in \mathbb{N}$ there exist $n \in \mathbb{N}$, $j_0, \dots, j_n \in \mathbb{N}$, such that $j_n = k$, $\alpha_{j_{l-1}, j_l} > 0$ for $\ell = 1, \dots, n$, $\alpha_{0, j_0} > 0$.

Let $\breve{x} \in \ell^{11}$ and $x(t) = S_1(t)\breve{x}$. Then the following hold.

(a) $\int_0^{\infty} |x_j(s)| ds < \infty$ *for all $j \in \mathbb{N}$.*

(b) If $\alpha_{00} = 0$ *and* $\check{x} \in \ell_+^{11}$, $x_0(t)$ *is monotone increasing in* $t \geq 0$.

(c) If $\limsup_{k\to\infty} \frac{1}{k} \sum_{j=1}^{\infty} j\alpha_{jk} < 0$, *then* $\int_0^\infty \left(\sum_{j=1}^{\infty} j|x_j(s)| \right) \mathrm{ds} < \infty$ *and* $\sum_{j=1}^{\infty} j|x_j(t)| \to 0$ *as* $t \to \infty$.

Proof of (a) (b): We can assume that $\alpha_{00} = 0$. For, if $\alpha_{00} < 0$, the solutions to (7) are dominated by those of the modified system where $\alpha_{00} = 0$ [46, 1.1]. Since $\alpha_{j0} = 0$ for all $j \in \mathbb{N}$, assumption (2) is valid in either case.

Let $\check{x} \in \ell_+^{11}$ and $x(t) = S_1(t)\check{x}$. By Theorem 1, $\|x(t)\| \leq \|\check{x}\|$ for all $t \geq 0$. By (22),

$$x_j(t) = \check{x}_j + \sum_{k=0}^{\infty} \alpha_{jk} \int_0^t x_k(s)\mathrm{ds}, \qquad j \in \mathbb{Z}_+, t \geq 0. \tag{25}$$

Since $x_j(t) \geq 0$ and $\alpha_{0k} \geq 0$, $x_0(t)$ is monotone increasing in $t \geq 0$.

Let $k \in \mathbb{N}$. Choose numbers $j_0, \ldots, j_n$ according to assumption (5).

Further, for all $j, k \in \mathbb{Z}_+$, $j \neq k$, and $t \geq 0$,

$$\|\check{x}\| \geq x_j(t) \geq -|\alpha_{jj}| \int_0^t x_j(s)\mathrm{ds} + \alpha_{jk} \int_0^t x_k(s)\mathrm{ds}.$$

For $j = 0, k = j_0$, since $\alpha_{00} = 0$,

$$\|\check{x}\| \geq \alpha_{0,j_0} \int_0^\infty x_{j_0}(s)\mathrm{ds}.$$

Step by step, $\int_0^\infty x_{j_l}(s)\mathrm{ds} < \infty$ for all $l = 0, \ldots, n$, in particular for $j_n = k$. Since every $\check{x} \in \ell^{11}$ can be represented as $\check{x} = \check{x}_+ - \check{x}_-$ with $\check{x}_\pm \in \ell_+^{11}$, $\int_0^\infty |x_j(s)|\mathrm{ds} < \infty$ for all $j \in \mathbb{N}$ and all $\check{x} \in \ell^{11}$.

(c) By Assumption (4) in Corollary 1, we can choose $m \in \mathbb{N}$ and $\epsilon_1 > 0$ such that Assumption 4 is satisfied. The integrability to infinity follows by integrating the inequality in Lemma 4. The convergence to 0 as $t \to \infty$ follows by applying Lebesgue's theorem of dominated convergence to the inequality in Lemma 4. Notice that $\xi_0 = 0$ because $\alpha_{j0} = 0$ for all $j \in \mathbb{N}$. $\square$

We use the last result to formulate conditions for semigroups to have a strictly negative growth bound (type). To this end consider the Banach sequence space

$$\tilde{\ell}^{11} = \left\{ y = (y_j)_{j=1}^{\infty};\ \|y\|^{\sim} = \sum_{j=1}^{\infty} j|y_j| < \infty \right\} \tag{26}$$

with norm $\| \sqcup \|^{\sim}$.

Corollary 2. *Let the assumptions of Proposition 2 be satisfied.*
Then the C_0-semigroup S_1 on ℓ^{11} satisfies

$$\sum_{j=1}^{\infty} j \left|[S_1(t)x]_j\right| \le M\mathrm{e}^{-\epsilon t} \sum_{j=1}^{\infty} j|x_j| \quad \forall x \in \ell^{11}$$

with appropriate $M, \epsilon > 0$. If in addition

$$\alpha_{00} < 0 \text{ and } \sup_{k\in\mathbb{N}} \frac{\alpha_{0k}}{k} < \infty,$$

then

$$\|S_1(t)\|_1 \le \tilde{M}\mathrm{e}^{-\tilde{\epsilon}t} \quad \forall t \ge 0,$$

with appropriate constants $\tilde{M} \ge 1$, $\tilde{\epsilon} > 0$.

Proof. Since $\alpha_{j0} = 0$ for all $j \in \mathbb{N}$, $[S_1(t)\breve{x}]_j$ does not depend on $\breve{x}_0$ for $j \in \mathbb{N}$. So we can define an operator family on $\tilde{\ell}^{11}$ by $[\tilde{S}(t)(\tilde{x}_k)_{k=1}^{\infty}]_j = [S_1(t)(\tilde{x}_k)_{k=0}^{\infty}]_j$, $j \in \mathbb{N}$, with an arbitrarily chosen $\tilde{x}_0$. Then

$$\left[\tilde{S}(t)\tilde{S}(r)\tilde{x}\right]_j = [S_1(t)S_1(r)(0,\tilde{x})]_j = [S_1(t+r)(0,\tilde{x})]_j = \left[\tilde{S}(t+r)\tilde{x}\right]_j.$$

So $\tilde{S}$ is a C_0-semigroup on $\tilde{\ell}^{11}$ and $\int_0^\infty \|\tilde{S}(t)\tilde{x}\|^{\sim}\mathrm{d}t < \infty$ for all $\tilde{x} \in \tilde{\ell}^{11}$ by Proposition 2 (c). By the Datko/Pazy theorem [34, Chapter 4, Theorem 4.1], there exist constants $\epsilon > 0$, $M \ge 1$ such that $\|\tilde{S}(t)\|^{\sim} \le M\mathrm{e}^{-\epsilon t}$ for all $t \ge 0$. This implies the first assertion. Let $x(t) = S_1(t)\breve{x}$. Then

$$x_0(t) = \breve{x}_0 + \sum_{k=0}^{\infty} \alpha_{0k} \int_0^t x_k(s)\mathrm{d}s.$$

By the triangle inequality and the additional assumptions,

$$x_0(t) \le |\breve{x}_0| - |\alpha_{00}| \int_0^t x_0(s)\mathrm{d}s + c\int_0^t \mathrm{e}^{-\epsilon s}\mathrm{d}s \sum_{j=1}^{\infty} j|\breve{x}_j|,$$

for some constant $c > 0$. By a Gronwall inequality, $x_0(t) \le M_0\|\breve{x}\|_1\mathrm{e}^{-\delta t}$ for some $M_0 \ge 1$ and $\delta > 0$, $\delta < |\alpha_{00}|, \epsilon$. In a similar way, we derive the same estimate for $-x_0(t)$. This implies the assertion. □

Obviously the semigroup $\tilde{S}$ in the previous proof is associated with the infinite matrix $(\alpha_{jk})_{j,k=1}^{\infty}$.

Proposition 3. *Let the coefficients α_{jk}, $j \in \mathbb{Z}_+$, $k \in \mathbb{N}$, satisfy the following assumptions:*

(1) $\alpha_{jk} \ge 0$ for $j \ne k$, $\alpha_{jj} \le 0$.

(2) $\sum_{j=0}^{\infty} \alpha_{jk} \leq 0$ *for all* $k \in \mathbb{N}$.

(3) There exist constants $c_0, c_1, \epsilon > 0$ *such that*

$$\sum_{j=1}^{\infty} j\alpha_{jk} \leq c_0 + c_1 k - \epsilon|\alpha_{kk}| \quad \forall k \in \mathbb{N}.$$

(4) For all $k \in \mathbb{N}$ *there exist* $n \in \mathbb{N}$, $j_0, \ldots, j_n \in \mathbb{N}$, *such that* $j_n = k$, $\alpha_{j_{l-1}, j_l} > 0$ *for* $\ell = 1, \ldots, n$, $\alpha_{0, j_0} > 0$.

Then the operator $\tilde{A}$ *with* $[\tilde{A}y]_j = \sum_{k=1}^{\infty} \alpha_{jk} y_k$ *with domain*

$$D(\tilde{A}) = \left\{ y \in \tilde{\ell}^{11}; \sum_{j=1}^{\infty} |\alpha_{jj}||y_j| < \infty, \tilde{A}y \in \tilde{\ell}^{11} \right\}$$

is the generator of a positive C_0*-semigroup* $\tilde{S}(t)$, $t \geq 0$, *on* $\tilde{\ell}^{11}$, *and there exist* $\epsilon > 0$, $M \geq 1$ *such that*

$$\|\tilde{S}(t)\|^{\sim} \leq M\mathrm{e}^{-\epsilon t} \quad \forall t \geq 0,$$

Proof. We set $\alpha_{j0} = 0$ for all $j \in \mathbb{Z}_+$ and obtain the result from the proof of Corollary 2. □

7. Essential growth bounds and asymptotic behavior

The (Kuratowski) measure of non-compactness [22, Sect. A.3.2, 18, Sect. 7.3, 39, Sect. 2.2] has the following sequential characterization in a metric space (X, d). It can be proved basically in the same way as the equivalence of sequential compactness on one hand and total boundedness and closedness on the other hand. If $Y \subseteq X$, the measure of noncompactness of Y, $\alpha(Y)$, equals

$$\alpha(Y) = \inf\left\{ c > 0;\ \text{each sequence } (x_n) \text{ in } Y \text{ has a subsequence } (x_{n_j}) \text{ with } \limsup_{j,k\to\infty} \mathrm{d}(x_{n_j}, x_{n_k}) \leq c \right\}. \tag{27}$$

Theorem 5. *Let the Assumptions 1 and 2 be satisfied and* $\sup_{k\in\mathbb{N}} \frac{\alpha_{jk}}{k} < \infty$ *for each* $j \in \mathbb{Z}_+$. *Then, for every bounded set* Y *in* ℓ^{11},

$$\alpha(S_1(t)Y) \leq \mathrm{e}^{\hat{\epsilon}t}\alpha(Y), \qquad \hat{\epsilon} = \limsup_{k\to\infty} \sum_{j=1}^{\infty} \frac{j\alpha_{jk}}{k}.$$

Proof. Let Y be a bounded subset of ℓ^{11}, $c > \alpha(Y)$. Fix $t \geq 0$ and let $(\tilde{x}_n)$ be a sequence in $S_1(t)Y$. Then $\tilde{x}_n = S_1(t)y_n$ with (y_n) being a sequence in Y. By (27), (y_n) has a subsequence (y_{n_j}) such that

$$\limsup_{j,k\to\infty} \|y_{n_j} - y_{n_k}\|_1 \leq c.$$

By Remark 1(22),

$$\left[S_1(t)y_{n_k}\right]_i - [S_1(r)y_{n_k}]_i = \sum_{j=0}^{\infty} \alpha_{ij} \int_r^t [S_1(s)y_{n_k}]_j \mathrm{d}s.$$

By our additional assumptions,

$$|(S_1(t)y_{n_k})_i - (S_1(r)y_{n_k})_i| \le \alpha_{i0} \int_r^t \|S_1(s)y_{n_k}\| \mathrm{d}s$$
$$+ \sup_{j\in\mathbb{N}} \frac{\alpha_{ij}}{j} \int_r^t \|S_1(s)y_{n_k}\|_1 \mathrm{d}s.$$

Since Y is a bounded set in ℓ^{11}, Theorem 4 implies that, for each $i \in \mathbb{Z}_+$, $\{(S_1(s)y_{n_k})_i; k \in \mathbb{N}\}$ is equi-continuous in $s \ge 0$. Similarly one shows that it is equi-bounded on every bounded interval. By the Arzela–Ascoli theorem, we can assume, after choosing another subsequence and a diagonalizaton procedure, that, for each $i \in \mathbb{Z}_+$, $(S_1(s)y_{n_k})_i$ converges as $k \to \infty$ uniformly in $s \in [0, t]$. Let $0 > -\epsilon_1 > \limsup_{k\to\infty} \sum_{j=1}^{\infty} \frac{\alpha_{jk}}{k}$. By Lemma 4, with $x(\cdot) = S_1(\cdot)y_{n_j} - S_1(\cdot)y_{n_k} = S_1(\cdot)(y_{n_j} - y_{n_k})$, for large enough $m \in \mathbb{N}$,

$$\begin{aligned}
&\|S_1(t)y_{n_j} - S_1(t)y_{n_k}\|_1 \\
&\le \sum_{i=0}^{m} \left|(S_1(t)y_{n_j})_i - (S_1(t)y_{n_k})_i\right| \\
&\quad + \left(\frac{1}{m}+1\right) \sum_{i=0}^{\infty} i \left|(S_1(t)y_{n_j})_i - (S_1(t)y_{n_k})_i\right| \\
&\le \left(1+\frac{1}{m}\right) \mathrm{e}^{-\epsilon_1 t} \|y_{n_j} - y_{n_k}\|_1 + \sum_{i=0}^{m} \left|(S_1(t)y_{n_j})_i - (S_1(t)y_{n_k})_i\right| \\
&\quad + \sum_{i=0}^{m} \left(\frac{1}{m}+1\right) \xi_i \int_0^t \mathrm{e}^{-\epsilon_1(t-s)} \left|(S_1(s)y_{n_j})_i - (S(s)y_{n_k})_i\right| \mathrm{d}s \qquad \forall t \ge 0,
\end{aligned}$$

with $\xi_i = \sum_{j=1}^{\infty} j\alpha_{ji} + \epsilon_1 i < \infty$. By our choice of subsequences, we have

$$\int_0^t \mathrm{e}^{-\epsilon_1(t-s)} \left|(S_1(s)y_{n_j})_i - (S(s)y_{n_k})_i\right| \mathrm{d}s \to 0, \qquad j,k\to\infty, \quad i \in \mathbb{Z}_+.$$

So

$$\begin{aligned}
&\limsup_{j,k\to\infty} \|S_1(t)y_{n_j} - S_1(t)y_{n_k}\|_1 \\
&\quad \le \left(1+\frac{1}{m}\right) \mathrm{e}^{-\epsilon_1 t} \limsup_{j,k\to\infty} \|y_{n_j} - y_{n_k}\|_1 \le \left(1+\frac{1}{m}\right) \mathrm{e}^{-\epsilon_1 t} c.
\end{aligned}$$

We take the limit $m \to \infty$,

$$\limsup_{j,k\to\infty} \|\tilde{x}_{n_j} - \tilde{x}_{n_k}\| \le \mathrm{e}^{-\epsilon_1 t} c.$$

By (27), since $(\tilde{x}_n)$ has been an arbitrary sequence in $S_1(t)Y$,

$$\alpha(S(t)Y) \le \mathrm{e}^{-\epsilon_1 t}\alpha(Y).$$

Since this holds for every $-\epsilon_1 > \limsup_{k\to\infty} \sum_{j=1}^{\infty} \frac{\alpha_{jk}}{k}$ and for every $\tilde{t} \ge 0$, the assertion follows. □

There are various equivalent characterizations of the essential type (essential growth bound) of an operator semigroup [11, Chapter IV, Definition 2.9] which are related to the various equivalent characterization of the essential spectral radius of a bounded linear operator [9, Sect. 9.8, 18, Lemma 2.3.3]. The one in [21, (8.6)] directly applies to our situation.

Corollary 3. *Let the assumptions of Theorem 5 be satisfied. Then the essential type (essential growth bound) of S_1 equals or is smaller than*

$$\limsup_{k\to\infty} \sum_{j=1}^{\infty} \frac{j\alpha_{jk}}{k}.$$

Theorem 6. *Let the coefficients α_{jk}, $j, k \in \mathbb{Z}_+$, satisfy the following assumptions:*

(a) $\alpha_{jk} \ge 0$ *if* $j \ne k$, $\alpha_{jj} \le 0$.

(b) $\displaystyle\sum_{j=0}^{\infty} \alpha_{jk} = 0 \qquad \forall k \in \mathbb{Z}_+.$

(c) There exists constants $c_0, c_1 > 0$, $\epsilon > 0$ *such that*

$$\sum_{j=1}^{\infty} j\alpha_{jk} \le c_0 + c_1 k - \epsilon|\alpha_{kk}| \qquad \forall k \in \mathbb{N}.$$

(d) $\displaystyle\limsup_{k\to\infty} \frac{1}{k} \sum_{j=1}^{\infty} j\alpha_{jk} < 0.$

(e) $\displaystyle\sup_{k=1}^{\infty} \frac{\alpha_{jk}}{k} < \infty$ *for all* $j \in \mathbb{Z}_+$.

Then $\sum_{j=0}^{\infty}[S_1(t)\breve{x}]_j = \sum_{j=0}^{\infty} \breve{x}_j$ *for all* $\breve{x} \in \ell^{11}_+$, $t \ge 0$. *Further there exist a non-zero positive linear operator P of finite rank on ℓ^{11} and some $\epsilon, M > 0$ such that $\|S_1(t) - P\|_1 \le M\mathrm{e}^{-\epsilon t}$ for all $t \ge 0$. P is a projection, $P^2 = P$, and maps ℓ^{11} into the null space of A_1. Further $\|P\breve{x}\| = \|\breve{x}\|$ for all $\breve{x} \in \ell^{11}_+$.*

Proof. By Lemma 3, v^* defined by $\langle x, v^*\rangle = \sum_{j=0}^{\infty} x_j$ is an element in $D(A_1^*)$ and $A_1^* v^* = 0$. In particular, 0 is a spectral value of A_1. Since $S_1(\cdot)$ is a bounded semigroup by Corollary 1, 0 is the spectral bound of A_1, $s(A_1)$.

Step 1: $0 = s(A_1)$ is a first order pole of $(\lambda - A_1)^{-1}$ with finite multiplicity.

By Corollary 3 and assumption (d), the essential growth bound of $S_1(\cdot)$ is less than 0. Then $0 = s(A_1)$ is not an essential spectral value of A_1 by [21, Proposition 8.6], and 0 is a pole of $(\lambda - A_1)^{-1}$ and the associated residue P has finite rank [21, Theorem A.3.3]. Moreover $P = \lim_{\lambda\to 0+} \lambda^p (\lambda - A_1)^{-1}$ with p being the order of the pole. Since $S_1(\cdot)$ is a positive semigroup, P is a positive operator and, as residue, not the 0 operator. Let $x \in \ell^{11}_+$. Then

$$\langle Px, v^*\rangle = \lim_{\lambda\to 0+} \lambda^p \langle(\lambda - A_1)^{-1}x, v^*\rangle = \lim_{\lambda\to 0+} \lambda^{p-1}\langle x, v^*\rangle.$$

If $p > 1$, $\langle Px, v^*\rangle = 0$. Since $Px \in \ell^{11}_+$, $Px = 0$. Since $x \in \ell^{11}_+$ has been arbitrary, $P = 0$, a contradiction. So 0 is a first order pole.

Step 2: Conclusion

Since the essential growth bound (or essential type) of $S_1(\cdot)$ is less than $0 = s(A_1)$ (which is also the type of $S_1(\cdot)$), we have $\omega_{ess}(A_1) < \omega_0(A_1)$ in the terminology of [23, Theorem 9.11] and our statement follows from its part (b). The additional properties of P follow from the fact that P is the limit of $S_1(t)$ as $t \to \infty$. □

In terms of population growth, Theorem 6 means that the probability distribution of the population size always converges toward a stationary distribution.

Corollary 4. *Let the assumptions of Theorem 6 be satisfied. Then every population size probability distribution $x(t)$ converges to a stationary distribution as $t \to \infty$. If $\alpha_{00} < 0$, the stationary limit distribution is not trivial, i.e., the expected population size has a non-zero limit,* $\lim_{t\to\infty} \sum_{j=1}^{\infty} j x_j(t) > 0$.

Proof. Let $\breve{x} \in \ell^{11}_+$, $\|\breve{x}\| = 1$, $x(t) = S_1(t)\breve{x}$. By Theorem 6, $\|x(t)\| = 1$ for all $t \geq 0$. Since $x(t) \to P\breve{x}$ in both ℓ^{11} and ℓ^1, $\|P\breve{x}\| = 1$. Since P maps into the null space of A_1,

$$0 = \sum_{k=0}^{\infty} \alpha_{0k}[P\breve{x}]_k.$$

Since $\alpha_{00} < 0$, $[P\breve{x}]_k > 0$ for some $k \in \mathbb{N}$. This implies the assertion. □

Without further assumptions, the limit distribution in Corollary 4 may depend on the initial size of the population. In the next theorem, the uniqueness of the limit distribution is enforced by irreducibility assumptions.

Definition 1. *The infinite matrix $(\alpha_{jk})_{j,k\in\mathbb{Z}_+}$ is called* irreducible *if, for every j, $k \in \mathbb{Z}_+$, $j \neq k$, there exists $n \in \mathbb{N}$ and $i_1, \dots, i_n \in \mathbb{Z}_+$ such that $i_1 = k$, $i_n = j$ and $\alpha_{i_{l+1},i_l} > 0$ for $l = 1, \dots, n-1$;*

If $k_0 \in \mathbb{N}$, the finite matrix $(\alpha_{jk})_{j,k=0}^{k_0}$ is called irreducible *if the analogous statement holds with the set $\mathbb{Z}$ be replaced by $\{0, \dots, k_0\}$.*

A number $k_0 \in \mathbb{N}$ is called the irreducibility bound of the infinite matrix (α_{jk}), if the matrix $(\alpha_{jk})_{j,k=0}^{k_0}$ is irreducible, $\alpha_{jk} = 0$ whenever $j > k_0$ and $k = 0, \dots, j-1$, and $\alpha_{kk} < 0$ for $k > k_0$.

Notice that the irreducibility together with the assumptions $\sum_{j=0}^{\infty} \alpha_{jk} \le 0$, $\alpha_{jk} \ge 0$ for $j \ne k$, implies that $\alpha_{kk} < 0$ for all $k \in \mathbb{Z}_+$. It is easy to see that the irreducibility bound (if there is one) is uniquely determined.

Theorem 7. *Let the coefficients α_{jk}, $j, k \in \mathbb{Z}_+$, satisfy the following assumptions:*

(a) $\alpha_{jk} \ge 0$ *if* $j \ne k$, $\alpha_{jj} \le 0$.

(b) $\displaystyle\sum_{j=0}^{\infty} \alpha_{jk} = 0 \qquad k \in \mathbb{Z}_+.$

(c) There exists constants $c_0, c_1 > 0$, $\epsilon > 0$ *such that*

$$\sum_{j=1}^{\infty} j\alpha_{jk} \le c_0 + c_1 k - \epsilon|\alpha_{kk}| \qquad \forall k \in \mathbb{N}.$$

(d) $\displaystyle\limsup_{k\to\infty} \frac{1}{k} \sum_{j=1}^{\infty} j\alpha_{jk} < 0.$

(e) $\displaystyle\sup_{k=1}^{\infty} \frac{\alpha_{jk}}{k} < \infty$ *for all* $j \in \mathbb{Z}_+$.

(f) Further assume that the infinite matrix $(\alpha_{jk})_{j,k=0}^{\infty}$ *is irreducible or has a irreducibility bound* $k_0 \in \mathbb{N}$.

Then there exists some $v \in \ell_+^{11}$, $v \ne 0$, *such that* $\sum_{k=0}^{\infty} \alpha_{jk} v_k = 0$ *for all* $j \in \mathbb{Z}_+$ *and*

$$S_1(t)x \to \frac{\sum_{j=0}^{\infty} x_j}{\sum_{j=0}^{\infty} v_j}\, v, \qquad t \to \infty \quad \forall x \in \ell^{11}.$$

v is uniquely determined up to a scalar factor. If $(\alpha_{jk})_{j,k=0}^{\infty}$ *is irreducible,* $v_j > 0$ *for all* $j \in \mathbb{Z}_+$*; if there is a irreducibility bound* $k_0 \in \mathbb{N}$, $v_j > 0$ *for* $j = 0, \ldots, k_0$ *and* $v_j = 0$ *for all* $j > k_0$.

Proof. If $(\alpha_{jk})_{j,k=0}^{\infty}$ is irreducible, the semigroup S_1 is irreducible and the assertion follows from [23, Theorem 9.11]. We assume that there is an irreducibility bound $k_0 \in \mathbb{N}$ for (α_{jk}) and show that the null space of A_1 is one-dimensional. Let v be an element in the null-space of A_1, $v \in \ell_+^{11}$.

Claim: $v_j = 0$ for all $j > k_0$.

Suppose $v_i > 0$ for some $i > k_0$. Since k_0 is an irreducibility bound and $A_1 v = 0$,

$$0 = \sum_{k=0}^{\infty} \alpha_{jk} v_k = \sum_{k=j}^{\infty} \alpha_{jk} v_k \qquad \forall j \ge i.$$

We add the last equations over j from i to ∞. Since $v \in D_0$, we can interchange the summation and

$$0 = \sum_{k=i}^{\infty} \left(\sum_{j=i}^{k} \alpha_{jk} \right) v_k.$$

Since $\alpha_{ii} < 0$ and $\sum_{j=i}^{k} \alpha_{jk} \le \sum_{j=0}^{\infty} \alpha_{jk} \le 0$ for all $k \ge i$,

$$0 = \alpha_{ii} v_i + \sum_{k=i+1}^{\infty} \left(\sum_{j=i}^{k} \alpha_{jk} \right) v_k \le \alpha_{ii} v_i < 0,$$

a contradiction. Hence $v_i = 0$.

We consider the subspace

$$X_0 = \{x = (x_j) \in \ell^{11}; x_j = 0, j > k_0\}.$$

It follows from assumption (f) that this subspace is invariant under S_1 and that the restriction of S_1 to X_0 is irreducible. By our claim, the null space of A_1 is contained in X_0 and is thus contained in the domain of the part of A_1 in X_0. This implies that the null space of A_1 is one-dimensional [21, Theorem 8.17] and spanned by a positive vector.

Since P maps into the one-dimensional null space of A_1 which is spanned by some $v \in \ell_+^{11}, \neq 0$, $P\breve{x} = \zeta v$ for some scalar ζ. By Theorem 6, $\zeta \sum_{j=0}^{\infty} \breve{v}_j = \sum_{j=0}^{\infty} (P\breve{x})_j = \sum_{j=0}^{\infty} \breve{x}_j$. We solve for ζ, $\zeta = \frac{\sum_{j=0}^{\infty} \breve{x}_j}{\sum_{j=0}^{\infty} \breve{v}_j}$. □

8. Continuous-time birth and death processes with immigration and catastrophes

If the size of a population is k, let β_k be the population birth rate, μ_k the population death rate, and ι_k the population immigration rate. We set $\eta_k = \beta_k + \iota_k$ with the understanding that $\beta_0 = 0$. A possible population emigration rate is absorbed in μ_k. We also allow catastrophes which wipe out the whole population, the associated rates are denotes by κ_k. The rates of transition from population size k to population size j, α_{jk}, are given by

$$\begin{cases} \alpha_{k+1,k} = \eta_k, & k \in \mathbb{Z}_+, \\ \alpha_{k-1,k} = \mu_k, & k \in \mathbb{N}, k \ge 2, \\ \alpha_{kk} = -(\eta_k + \mu_k + \kappa_k), & k \in \mathbb{N}, \\ \alpha_{00} = -\eta_0 = -\iota_0, & \\ \alpha_{0k} = \kappa_k, & k \in \mathbb{N}, k \ge 2, \\ \alpha_{01} = \kappa_1 + \mu_1, & \\ \alpha_{jk} = 0, & \text{otherwise}. \end{cases} \tag{28}$$

Then

$$\sum_{j=0}^{\infty} \alpha_{jk} = 0, \qquad k = 0, 1, \ldots. \tag{29}$$

and

$$\sum_{j=1}^{\infty} j\alpha_{jk} = (k+1)\eta_k + (k-1)\mu_k - k(\eta_k + \mu_k + \kappa_k), \quad k \in \mathbb{N},$$

$$\sum_{j=1}^{\infty} j\alpha_{j0} = \eta_0.$$

If we define $\mu_0 = 0 = \kappa_0$,

$$\sum_{j=1}^{\infty} j\alpha_{jk} = \eta_k - \mu_k - k\kappa_k, \quad k \in \mathbb{Z}_+. \tag{30}$$

Obviously the Assumptions 1 are satisfied with $\alpha^\diamond = 0$. There are many possible assumptions which imply Assumption 2. The following one has been chosen for brevity, not generality.

Assumption 5. (a) $\eta_n, \kappa_n \geq 0$, $\mu_n > 0$ for all $n \in \mathbb{N}$,
(b) $\sup_{k \geq 1} \frac{2\eta_k - \mu_k}{k} < \infty$.

Theorem 8. *Let the Assumptions 5 be satisfied. Then there exists a C_0-semigroup S_1 on ℓ^{11} such that $x(t) = S_1(t)\breve{x}$ is the unique continuous solution $x : \mathbb{R}_+ \to \ell^{11}$ of*

$$\begin{aligned}
x_0(t) &= \breve{x}_0 + \mu_1 \int_0^t x_1(s)\mathrm{ds} + \sum_{j=1}^{\infty} \kappa_j \int_0^t x_j(s)\mathrm{ds} - \iota_0 \int_0^t x_0(s)\mathrm{ds},\\
x_1' &= \iota_0 x_0 + \mu_2 x_2 - (\iota_1 + \beta_1 + \mu_1 + \kappa_1)x_1, \qquad x_1(0) = \breve{x}_1,\\
x_j' &= (\beta_{j-1} + \iota_{j-1})x_{j-1} + \mu_{j+1}x_{j+1} - (\iota_j + \beta_j + \mu_j + \kappa_j)x_j,\\
x_j(0) &= \breve{x}_j,
\end{aligned}$$

with the property that $\sum_{j=1}^{\infty}(\beta_j + \iota_j + \mu_j + \kappa_j)\left|\int_0^t x_j(s)\mathrm{ds}\right| < \infty$ for all $t \geq 0$.

Further $\sum_{j=0}^{\infty} x_j(t) = \sum_{j=0}^{\infty} \breve{x}_j$ for all $t \geq 0$.

Proof. By (30), for $k \in \mathbb{N}$,

$$\sum_{j=1}^{\infty} j\alpha_{jk} \leq \left(1 + \frac{1}{3}\right)\eta_k - \left(1 - \frac{1}{3}\right)\mu_k - \frac{1}{3}|\alpha_{kk}| = \frac{2}{3}(2\eta_k - \mu_k) - \frac{1}{3}|\alpha_{kk}|.$$

Further, again by (30) and by $\alpha_{00} = -\eta_0$,

$$\sum_{j=1}^{\infty} j\alpha_{j0} = \eta_0 = \frac{4}{3}\eta_0 - \frac{1}{3}|\alpha_{00}|.$$

So Assumption 2 is satisfied with $\epsilon = 1/3$, $c_0 = (4/3)\eta_0$, and $c_1 = (2/3)\sup_{k \in \mathbb{N}} \frac{2\eta_k - \mu_k}{k}$. By Theorem 2, the part in ℓ^{11} of the operator A in (18) generates a C_0-semigroup S_1 on ℓ^{11}. If $x(t) = S_1(t)\breve{x}$, $x_j(t)$ satisfies (22). Notice that, in this case, the equations can be differentiated for $j \in \mathbb{N}$. The last statement follows from Lemma 3. $\square$

In order to motivate some stronger assumptions than Assumption 5, we revisit the simple birth and death process where $\iota_n = 0 = \kappa_n$ for all $n \in \mathbb{Z}$ and $\beta_n = n\beta_1$, $\mu_n = n\mu_1$. Let $x(t) = S_1(t)\breve{x}$ be the probability distribution of population size. Assume that the initial population has size N, $\breve{x} = (\delta_{Nk})$. By Proposition 1, $x_0(t)$

is monotone increasing in $t \geq 0$ and $x_j(t) \to 0$ as $t \to \infty$ for all $j \in \mathbb{N}$. Let $\beta_1 > \mu_1$. Then $\lim_{t\to\infty} x_0(t) = \left(\frac{\mu_1}{\beta_1}\right)^N < 1$ [1, (6.8)]. However, $\sum_{j=0}^{\infty} x_j(t) = 1$ for all $t \geq 0$. So, for each $k \in \mathbb{N}$,

$$\sum_{j=k}^{\infty} x_j(t) \to 1 - \left(\frac{\mu_1}{\beta_1}\right)^N > 0, \qquad t \to \infty.$$

Further $\sum_{j=1}^{\infty} j x_j(t) = \mathrm{e}^{(\beta_1-\mu_1)t} N \to \infty$ as $t \to \infty$ [1, Table 6.1].

Motivated by this example, we make the following assumptions.

Assumption 6. (a) $\eta_n, \kappa_n \geq 0$, $\mu_n > 0$ for all $n \in \mathbb{N}$,
(b) $\inf_{n=1}^{\infty} \frac{\mu_n}{n} > 0, \qquad \limsup_{n\to\infty} \frac{\eta_n}{\mu_n} < 1$.
(c) $\sup_{n=1}^{\infty} \frac{\kappa_n}{n} < \infty$.

Under these assumptions, the Assumptions 1, 2 and 4 are satisfied. Further $\sup_{k=1}^{\infty} \frac{\alpha_{jk}}{k} < \infty$ for all $j \in \mathbb{Z}_+$.

8.1. Population extinction

The population goes extinct without immigration.

Theorem 9. *Let the Assumptions 6 be satisfied. If $\iota_0 = 0$, then there exists some $\epsilon > 0$ such that*

$$\sum_{j=1}^{\infty} j|x_j(t)|\mathrm{dt} \leq M\mathrm{e}^{-\epsilon t} \sum_{j=1}^{\infty} (1+j)|\breve{x}_j|$$

for all solutions x in the sense of Theorem 8.

8.2. Population survival and asymptotic behavior

We now assume that an extinct population can be resurrected by immigration, i.e., $\iota_0 > 0$. Since this means that $\alpha_{00} < 0$, we obtain the following result from Corollary 4.

Theorem 10. *Let the Assumptions 6 be satisfied and $\iota_0 > 0$. Then every population size probability distribution $x(t)$ converges to a stationary distribution as $t \to \infty$. The stationary limit distribution is not trivial, i.e., the expected population size has a non-zero limit,* $\lim_{t\to\infty} \sum_{j=1}^{\infty} j x_j(t) > 0$.

We now assume that either births and immigration do not completely stop however large the population or that there is a birth and immigration threshold $k_0 \in \mathbb{N}$ at which births and immigration stop, i.e. $\eta_k > 0$ for $k = 0, \ldots, k_0 - 1$ and $\eta_k = 0$ for all $k \geq k_0$.

Theorem 11. *Assume that the Assumptions 6 are satisfied. Further assume that the coefficients $\eta_k = \beta_k + \iota_k$ satisfy one of the following two assumptions:*

(i) $\eta_k > 0$ *for all* $k \in Z_+$.
(ii) There exists some $k_0 \in \mathbb{N}$ *such that* $\eta_k > 0$ *for* $k = 0, \dots, k_0 - 1$ *and* $\eta_k = 0$ *for* $k \geq k_0$.

Then there exists some $v \in \ell_+^{11}$, $v \neq 0$, *such that*

$$S_1(t)x \to \frac{\sum_{j=0}^{\infty} x_j}{\sum_{j=0}^{\infty} v_j} v, \quad t \to \infty \quad \forall x \in \ell^{11}.$$

v *is uniquely determined up to a scalar factor. If (i) holds,* $v_j > 0$ *for all* $j \in \mathbb{Z}_+$*; if (ii) holds,* $v_j > 0$ *for* $j = 0, \dots, k_0$ *and* $v_j = 0$ *for all* $j > k_0$.

In terms of the probability distributions of population size this means the following: There exists a unique stationary probability distribution v (with one of the two properties spelt out in Theorem 11) such that $x(t) \to v$ for all probability distributions $x(\cdot)$.

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