University of Florida

MMB | **Practice Final** | **Due: Not due**

Name:

ID #:

Instructor:

Directions: You have until 5:00 p.m. on the due date to answer the following questions. You must show all your work as neatly and clearly as possible and indicate the final answer clearly. You may use any books and you can work together but each of you must submit a homework.

Problem	Possible	Points
1	5	
2	5	
All	10	
Total	20	

(1) The following model has been proposed to study the interactions between baleen whales and their main food source, krill in the southern ocean:

$$\begin{aligned} x'(t) &= rx\left(1 - \frac{x}{K}\right) - axy \\ y'(t) &= sy\left(1 - \frac{y}{bx}\right) \end{aligned} \tag{1}$$

where r, b, s, K are positive constants (parameters). Here the whale carrying capacity is not constant but a function of the krill population

$$K_w = bx.$$

(a) Determine the model equilibria (steady states) and the conditions for their existence.

(b) Determine the stability of each equilibrium.

(c) Draw a phase-plane diagram with the nullclines and the direction of the vector field along the nullclines.

(2) Consider the following system

$$x' = a_1 x^2 - a_2 x^3 - bxy$$
$$y' = dxy - \gamma y^2 \tag{2}$$

where a_1, a_2, b, d and γ are positive constants.

(a) Find the equations of the nullclines. Sketch the nullclines in the (x, y) plane. Sketch the direction field along the nullclines.

(b) Find all non-negative equilibria of model (2) and the conditions under which they exist.

(c) With the following values of the parameters, determine the stability of each equilibrium and the type (spiral, node, ect).

$$a_1 = 1, \qquad a_2 = 0.05, \qquad b = 5, \qquad d = 1, \qquad \gamma = 10.$$

(3) Consider the single species population model exhibiting the Allee effect in which the per capita growth rate is a quadratic function of the density and is subject to time delays:

$$x'(t) = x(t)[a + bx(t-\tau) - cx^2(t-\tau)].$$

where a, b and c are positive parameters.

(a) Determine the equilibria of the model.

(b) Linearize the equation around the non-trivial equilibrium. Simplify as much as possible.

(c) Derive the characteristic equation. Separate the real and the imaginary part.

(d) Derive a condition on the delay that guarantees the local stability of the non-trivial equilibrium.

(4) Consider the following discrete single species population model

$$x_{n+1} = \left[a_1 + \frac{a_2}{1 + e^{c(x_n - b)}}\right] x_n$$

where a, b and c are positive parameters.

(a) Determine the equilibria of the model and the conditions on the parameters that guarantee their existence.

(b) Linearize the equation around the non-trivial equilibrium. Simplify as much as possible.

(c) Determine the stability of the trivial equilibrium.

(d) Derive a condition that guarantees the local stability of the non-trivial equilibrium.

(5) Consider the epidemic model

$$
\begin{aligned}
S'(t) &= \Lambda - \frac{\beta SI}{A+S} - \mu S \\
I'(t) &= \frac{\beta SI}{A+S} - (\mu + \gamma) I
\end{aligned}
\tag{3}
$$

where S is the number of susceptible individuals and I is the number of infected individuals. μ is the natural death rate, γ is the recovery rate, β is the transmission rate, and Λ is the recruitment rate.

(a) Compute the disease-free equilibrium. Compute the Jacobian around the disease-free equilibrium. Compute the reproduction number $\mathcal{R}_0$. Show that the disease-free equilibrium is l.a.s if $\mathcal{R}_0 < 1$ and unstable otherwise.

(b) Compute the endemic equilibrium.

(c) Show that Hopf bifurcation may occur.

(6) Consider the age-structured population model.

$$\begin{aligned} &\frac{\partial u}{\partial t} + \frac{\partial u}{\partial t} = -\mu(a)u \\ &u(t,0) = \int_0^\infty \beta(a)u(t,a)da \\ &u(0,a) = u_0(a) \end{aligned} \tag{4}$$

Assume $\mu(a) = \mu$, where μ is a constant. Furthermore, assume that

$$\beta(a) = \begin{cases} 0 & 0 \le a < A \\ \bar{\beta}e^{-\gamma a} & a > A \end{cases} \tag{5}$$

where $\bar{\beta}$, γ and A are constants.

(a) Compute the growth rate of the population $\bar{\lambda}$.

(b) Compute the net reproduction rate of the population $\mathcal{R}$.

(c) Interpret the biological meaning of A. How does A affect $\bar{\lambda}$ and $\mathcal{R}$? Why does that make sense biologically?

(7) The following model has been proposed to model an infectious disease with saturating treatment:

$$I'(t) = \beta I(N - I) - \frac{\alpha I}{A + I} \tag{6}$$

where β, α and A are parameters and N is the constant total population size.

(a) Compute the disease-free equilibrium and the model reproduction number $\mathcal{R}_0$.

(b) Show that if $\mathcal{R}_0 > 1$ there is a unique endemic equilibrium.

(c) Show that if $\mathcal{R}_0 < 1$ there may be zero endemic equilibria or two endemic equilibria. What condition(s) guarantee the presence of two equilibria?

(8) Suppose fishing is regulated within a zone of L kilometers from a country's shore (on a straight line), but outside of this zone overfishing is so extreme that the population is zero. Assume that fish reproduction follows logistic law in the regulated zone, and fish are harvested with a constant effort E. Then we have the following model for the fish population $N(t, x)$ for time t and position x from the shore:

$$
\begin{aligned}
&\frac{\partial N}{\partial t} = rN\left(1 - \frac{N}{K}\right) - EN + D\frac{\partial^2 N}{\partial x^2} \\
&N(t, L) = 0, \quad \frac{\partial N}{\partial x}(t, 0) = 0 \\
&N(0, x) = N_0(x)
\end{aligned}
\tag{7}
$$

where $K, E, D > 0$ and $r > E$.

(a) Show that $N^* = 0$ is a constant solution of the problem above.

(b) Linearize the differential equation about the solution $N_1* = 0$ with $v = N + 0$ and show that the linearization satisfies the equation

$$\frac{\partial v}{\partial t} = (r - E)v + D\frac{\partial^2 v}{\partial x^2}.$$

(c) Solutions to the linearized equation in $v(t, x)$ are sums of terms of the form

$$v_n(t, x) = e^{\sigma t}\cos(k_n x), \qquad k_n = \frac{(2n+1)\pi}{2L}, \quad n = 0, 1, 2, \ldots,$$

and

$$v = \sum_n v_n$$

where σ depends on n.

(d) If $\sigma < 0$ then $v(t, x) \to 0$ as $t \to \infty$ that is the zero equilibrium is locally stable and the fish stock collapses. To prevent the fish stock from collapsing we need $\sigma \geq 0$. Derive the expression for σ and from there show that to prevent the fish stock from collapsing the zone must satisfy

$$L \geq \pi\sqrt{D/(4(r - E))}.$$

(9) Consider the SIS epidemic model with two strains:

$$\begin{aligned} S'(t) &= \Lambda - \beta_1 S I_1 - \beta_2 S I_2 - \mu S + \gamma_1 I_1 + \gamma_2 I_2 \\ I_1'(t) &= \beta_1 S I_1 - (\mu + \gamma_1) I_1 \\ I_2'(t) &= \beta_2 S I_2 - (\mu + \gamma_2) I_2 \end{aligned} \tag{8}$$

where S is the number of susceptible individuals and I_1 is the number of individuals infected with strain one while I_2 is the number of individuals infected with strain two. Furthermore, μ is the natural death rate, γ_1 and γ_2 are the recovery rates, β_1 and β_2 are the transmission rates, and Λ is the recruitment rate.

(a) The total population size is $N(t) = S(t) + I_1(t) + I_2(t)$. Notice that $N'(t) = 0$. hence the total population size is constant. Say N. Show that we can express the number of susceptible individuals in terms of I_1 and I_2 and eliminate S from the last two equations, thus obtaining the system

$$\begin{aligned} I_1'(t) &= \beta_1 (N - I_1 - I_2) I_1 - (\mu + \gamma_1) I_1 \\ I_2'(t) &= \beta_2 (N - I_1 - I_2) I_2 - (\mu + \gamma_2) I_2 \end{aligned} \tag{9}$$

(b) What is the disease-free equilibrium of system (9)? Find the Jacobian around the disease-free equilibrium. Find the conditions which give local stability of the disease-free equilibrium. These conditions define two reproduction numbers – the reproduction number of strain one $\mathcal{R}_1$ and the reproduction number of strain two $\mathcal{R}_2$. Find the reproduction numbers.

(c) For system (9) find the two semi-trivial equilibria. Show that equilibrium of strain i exists iff $\mathcal{R}_i > 1$.

(d) If $\mathcal{R}_1 \neq \mathcal{R}_2$, is there an equilibrium in which both strains coexist for system (9)?

(e) Find the local stability of the semi-trivial equilibria of system (9).

(10) Consider the following SIS epidemic model with disease-induced mortality γ and saturating incidence:

$$\begin{aligned} S' &= \Lambda - \frac{\beta IS}{A+I} + \alpha I - \mu S \\ I' &= \frac{\beta IS}{A+I} - (\alpha+\gamma+\mu)I \end{aligned} \tag{10}$$

where S is the number of susceptibles, I is the number of infected, β is the transmission rate, α is the recovery rate, Λ is the birth rate, μ is the natural death rate. Use Dulac's criterium to rule out oscillations.